Electronics for Physicists

Thomas Flanagan
Max Robinson

Contents

Chapter 1

Electrical Fundamentals

1.1 Fundamental Electrical Quantities

1.1.1 Electric Charge

Electric charge is a fundamental property of matter which determines how said matter responds to electromagnetic forces. The standard symbol for charge is q. If the charge is constant, Q is often used. Q is often used also to indicate the *magnitude* of a time-varying charge (e.g. if the charge on an object is sinusoidal, it may be expressed as $q(t) = Q\sin(\omega t)$. We'll see more expressions like this later.

> **Note**
>
> This convention is quite common. A time-varying quantity is indicated by a lowercase letter. A constant quantity, or the (constant) *amplitude* of a time-varying quantity, is indicated by the capital version of the same letter.

Probably your first charged experience involved static electricity. You walk across the carpet on a cold winter morning, reach for the doorknob and—ZAP—you get an electric shock. Your body acquired electric charge by friction with the carpet.

Electric charge can be placed on objects under somewhat more controlled circumstances. One way is to rub a rubber rod with a piece of cat's fur. The rod will now attract small objects such as bits of paper. The charge on the rod induces the opposite charge on the bits of paper. The fur will also attract small objects but, it is not as easy to handle as the rod.

The property of electric charge will cause two bodies which possess it to exert forces on one another. The magnitude of this force is directly proportional to the amount of charge on each body and inversely proportional to the square of the distance between the two bodies. This is called Coulomb's law, and it can be expressed mathematically as

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2},$$

where F is the force, q_1 is the amount of charge on body 1, q_2 is the amount of charge on body 2, and r is the distance between the two bodies. If the two bodies have charges of the same sign, the force is repulsive. If the two bodies have charges of opposite sign, the force is attractive.

If the distance is one meter, the force is one newton, and the two bodies have the same amount of charge, the amount of charge on each body is defined to be one coulomb (symbol "C"), which is the standard unit of charge in the MKS system. The charge on one electron is $-1.601\,864 \times 10^{-19}$ C. Therefore, one coulomb of charge is equal to the absolute value of the total charge of 6.24×10^{18} electrons.

1.1.2 Electric Potential

Potential energy (usually denoted U), is loosely defined as the ability to do work, and there is a potential associated with every conservative force. Examples from mechanics include gravitational potential energy $U_g = mgh$ and elastic potential energy $U_{sp} = \frac{1}{2}kx^2$. Electrical potential energy is also an expression of work which can be done or energy stored. The units of potential energy are joules.

Electric potential is potential energy per charge, and its typical symbol is v or V. The potential energy of some charge q at a point where the electric potential is v is $U = qv$, so electric potential can be written

$$v = \frac{U}{q}.$$

Often in electronics, we prefer to think about work done w done on charge than about potential energy. By conservation of energy, the change in potential energy is equal to the work, so the above equation is usually expressed as

$$v = \frac{w}{q} \tag{1.1}$$

Taking potential energy *per charge* is useful because it is simply a function of position in the circuit.

Electrical potential, like potential energy, has no absolute zero. Electrical potential is always expressed as a potential *difference* between two points. In some cases, one of the points is implied or understood in context. Electrical potential is meaningless unless two points are specified, implied, or understood.

Because potential is specified between two points, the term *potential difference* is often used. The "difference" is usually omitted when speaking and often omitted when writing; "potential difference" is always understood by speaker, listener, writer, and reader.

The unit of electrical potential difference is the volt (symbol "V"), defined as

$$\text{volt} = \frac{\text{joule}}{\text{coulomb}}. \tag{1.2}$$

From the unit, potential difference is often called the voltage difference or the voltage drop, but most often just the voltage. As with units of current, the mechanical units of charge could be substituted into Eq. (1.2) to obtain purely mechanical units of voltage difference.

If work is done on a given amount of charge, the potential increases. If the charge does work, the potential decreases. If one joule of work is done on one coulomb of charge, the charge is moved through a potential difference of +1 V. If one coulomb of charge does one joule of work, the charge has moved through a potential difference of −1 V.

The term "electromotive force" was once regularly used to describe electrical potential difference. This led to the use of the symbol emf and later to the letter $\mathcal{E}$ to symbolize potential difference. In more recent times, the symbol V has been adopted for voltage, though $\mathcal{E}$ is still often used for the voltage of energy *sources*.

In most "at-home" circuits, voltages will be between a few mV and a few tens of V. Wall outlets deliver 120 V[1]. You may, at some point, need to devise circuitry to decrease this to a more manageable value, which we'll discuss later.

[1] This is true in North America, and it varies from country to country. In Europe, it is usually either 220 V or 230 V.

Note

You may have noticed that there was no discussion of kinetic energy above and asked, "Isn't the external work the *sum* of changes in potential *and* kinetic energy?" Yes, but in electronics, we will practically never be concerned with kinetic energy $K = mv^2/2$. The reason is that electrons (1) have a very tiny mass $m_e = 9.11 \times 10^{-31}$ kg and (2) tend to move very slowly through conductors, for reasons that you will learn if you take a course in condensed matter physics (if you will not take such a course and want to know why, do an internet search for "Drude model").

The upshot is that the potential energy of an electron at a potential of 1 V is $U = 1.602 \times 10^{-19}$ J, while a typical kinetic energy for an electron in a metal is 10^{-40} J. Kinetic energy can be ignored.

1.1.3 Electric Current

Electric current is the motion of electric charge through any conducting material. Mathematically, it defined as the amount of charge q flowing past some particular point or into or out of some circuit element per time. Its usual symbol is i or I, and it is given by

$$i \equiv \frac{dq}{dt}. \tag{1.3}$$

An alternate way of defining current would be to specify the number of electrons per second passing a point. Although this would be a perfectly valid way of specifying the amount of current in a conductor, the numbers which would result in practical applications would be so large as to be unwieldy. More practical numbers result if the coulomb is taken as the basic unit of charge instead of the electron. The most commonly used unit of current is the ampere (symbol "A"). One ampere is equal to one coulomb of charge per second passing a point on a conductor:

$$\text{ampere} = \frac{\text{coulomb}}{\text{second}}.$$

If desired, the mechanical units of charge could be substituted into this equation, and a strictly mechanical definition of current could be obtained. This is left as an exercise at the end of this chapter.

When an electric current moves through a circuit, the negatively charged electrons flow toward the positively charged part of the circuit. This means that electrons come out of the − side of a battery and go into the + side.

Physicists and electrical engineers alike prefer to use **conventional current** instead of electron current. Conventional current does what you would expect it to do. It flows out of the + side of a battery and flows into the − side. In other words, it flows from high electric potential to low potential.

Some students find this confusing. If you are among the confused, try this. Never think about electron current. Banish electron current forever from your mind and think only in terms of conventional current. If any one man can be blamed for this confusion it seems to be the fault of Benjamin Franklin. Everyone knows about his "kite in the lightning storm" experiment. What few people know is that Mr. Franklin wrote one of the first—if not the first—textbook on electricity. He assembled all that was known at that time about electricity and put it in one book. He seems to have added very little in the way of original work himself, but one thing he did add was a sign convention for electric charge. The convention he chose was a guess and he guessed wrong. Just think of it, if he had guessed the other way, we would have positive electrons and conventional current would be the same as electron current.

Typical currents in "at-home" circuits are a few mA. You can feel 1 mA, the maximum harmless current for the human body is accepted as being around 5 mA. Electrical currents become dangerous (with extended contact) around 100 mA. About 200 mA is the most a typical breadboard can handle without sustaining damage.

1.1.4 Electrical Power

In mechanics, power is the rate at which work is done, or work per unit time,

$$p = \frac{dw}{dt}. \tag{1.4}$$

Electrical power is the same. We can write it in terms of electrical quantities. By the chain rule,

$$p = \frac{dw}{dq}\frac{dq}{dt}.$$

Substitution of Eq. (1.1) and Eq. (1.3) yields

$$p = vi. \tag{1.5}$$

If we substitute the mechanical units of voltage and current into Eq. (1.5), we have the watt (symbol "W"):

$$\text{watt} = \frac{\text{joule}}{\text{coulomb}}\frac{\text{coulomb}}{\text{second}} = \frac{\text{joule}}{\text{second}},$$

as expected from mechanics.

1.2 Circuit Terminology

We will now define some ubiquitous terms in circuit analysis.

Circuit element Any device which obeys the lumped element discipline (see the next section).
Circuit A network of circuit elements joined at their terminals.
Node A (possibly extended) point at which two or more circuit elements meet.
Branch A path from one node to another.
Loop A closed path around some set of elements in a circuit.
Mesh A loop which has no other loops inside it.
Ground An arbitrarily chosen point in a circuit assigned to be at zero electric potential. Any point can be chosen, technically, though some will make the analysis easier.

Next, what does it mean to solve a circuit? We can consider a circuit to be *solved* when we know the current through and the voltage across every device about which we care. Figuring out these **node voltages** and **branch currents** is what it means to solve a circuit. Note that we talk about the *voltage* at a *node* because a node has no circuit elements within it, since nodes by definition are separated by circuit elements, and potential only changes *across* circuit elements. We talk about the *current* through a *branch* because branches separate nodes, and the node voltages are what drive currents.

1.3 The Lumped Matter Discipline and Kirchhoff's Laws

The lumped matter discipline is basically a set of constraints placed on electrical components to make circuit analysis simpler—to enable us to solve circuits without having to pull out the full machinery of Maxwell's equations.

The constraints are as follows.

1. The charge in a conductor is constant in time:
$$\frac{\partial q}{\partial t} = 0.$$
2. The magnetic flux outside of a conductor is constant in time:
$$\frac{\partial \phi_{\mathrm{B}}}{\partial t} = 0.$$
3. All signals in the circuit propagate significantly slower than the speed of light through the circuit.

The first constraint means that the charge flowing into some element equals the charge flowing out, and, therefore, that the current flowing into the element equals the current flowing out. Due to the repulsion of like charges, this will be true on any timescales in which we are interested here.

The result is **Kirchhoff's current law (KCL)**: The algebraic sum of all currents flowing into or out of any point or any element is zero:

$$\sum_n (i_n)_{\mathrm{in}} = \sum_n (i_n)_{\mathrm{out}} = 0. \tag{1.6}$$

The second constraint allows us to neglect Faraday's law,

$$\mathcal{E} = -\frac{d\phi_{\mathrm{B}}}{dt},$$

outside of circuit elements. This allows us to define a unique electric potential (with respect to some reference point) at every point in a circuit. It means that potential changes *only* across circuit elements, and not across wires.

The result is **Kirchhoff's voltage law (KVL)**: The algebraic sum of all potential drops or of all potential rises around any closed path in a circuit is zero:

$$\sum_n v_n = 0. \tag{1.7}$$

Equivalently, the voltage between any two nodes is independent of the path taken between them.

1.4 Primary Circuit Elements

In order for electricity to be useful, it must be harnessed to do work for us. The usual way of making electricity do work is to connect a source of electric energy to a device which converts the electric energy into another useful form of energy. An example of this is the flashlight, the schematic diagram of which is shown in Fig. 1.1. The battery on the left converts chemical energy into electrical energy. The lamp on the right converts the electrical energy into electromagnetic energy, some of which is in the visible light part of

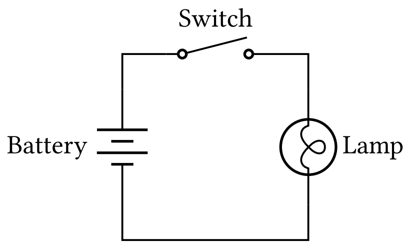


Figure 1.1: A schematic diagram of a flashlight.

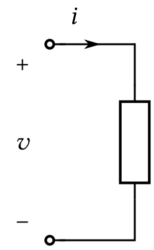


Figure 1.2: A generic circuit element to show the passive sign convention.

the electromagnetic spectrum. The switch is used to interrupt the flow of electrons and permit the light to be turned on and off. The lines on the drawing indicate conductors such as wire which normally have negligible resistance compared to the rest of the circuit.

Since our goal in solving a circuit is to determine node voltages and branch currents, we want to describe each circuit element in terms of its **element relation**, which is an expression which relates the voltage across that element to the current through it. This is also called an element's "$i - v$ characteristic."

1.4.1 Passive Sign Convention and Power

Consider a generic circuit element as shown in Fig. 1.2. Note how both the sign of the voltage and the direction of the current are indicated. They illustrate the **passive sign convention**. The + and − indicate the *polarity* of the voltage. As we move across an element, if we go from a + to a − in voltage, this is a *voltage drop*. If we go from − to +, it is a voltage rise. The branch current between any two nodes points from + voltage to − voltage.

> **Important**
> While the passive sign convention is not the only convention, it is common, and we will use it throughout this book. The most important thing is to use *only one* sign convention throughout a problem. Otherwise, you *will* make sign errors.

Recall electric power from Eq. (1.5):

$$p = iv.$$

But there's a question remaining: is that the power *absorbed* by the element or the power *delivered* by the element? The answer is that as long as the passive sign convention is followed, the above equation gives the power *absorbed* by the element:

$$p_\mathrm{a} = iv. \tag{1.8}$$

The power that the element delivers to the circuit is the opposite:

$$p_\mathrm{d} = -iv. \tag{1.9}$$

1.4.2 Ideal Sources and Wires

The first elements we shall describe are ideal sources.

An **ideal, independent voltage source**, as shown in Fig. 1.3(a), is designed to maintain some specified voltage (v_s in the figure) across its terminals, and the current through it will be whatever it needs to be to satisfy the circuitry attached to its terminals. The typical example of a voltage source is a battery (as in Fig. 1.1). Batteries, however, are not *ideal* voltage sources, as we'll see later[2]. The short explanation is that batteries have some internal resistance which, in some circumstances, may noticeably affect the voltage they are supposed to deliver. Much better are "dc regulated power supplies" found in most electronics labs, though even these are not *perfectly* ideal.

An **ideal, independent current source**, like that in Fig. 1.3(b), requires a specified current i_s, and the voltage across its terminals is determined by the rest of the circuit (not shown). The arrow inside the symbol for the current source indicates the direction of the current enforced by the source. As drawn, the arrows for i and i_s are in opposite directions, so one must be negative relative to the other. The power absorbed by this element is this $p_a = iv = -i_s v$. Consequently, if i_s and v are both positive numbers, the power absorbed by the source is negative (or, equivalently, power is delivered by the device, rather than absorbed by it). We'll clarify this with an example once we have introduced resistors in the next section.

Fig. 1.3(c) shows an **ideal open circuit**, or a gap or break in the circuit. The voltage across the terminals can be whatever the external circuit requires, but the current through it is always zero. The opposite of this is the **ideal short circuit**, or **ideal wire**, shown in Fig. 1.3(d). Ideal wire drops no voltage regardless of the current through it. Unless otherwise specified, all wires in circuit diagrams are treated as ideal. In the rare case that the resistance of the wire matters, it will be "lumped" into one or more discrete resistors.

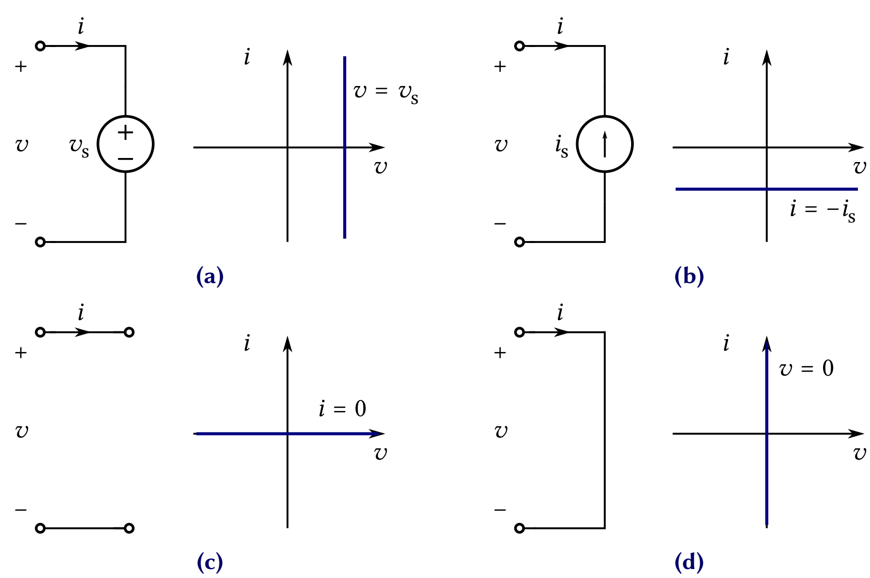
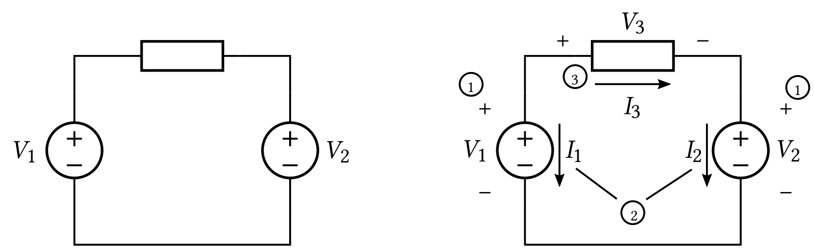


Figure 1.3: The circuit symbols and $i - v$ characteristics for the simplest four circuit elements: (a) an ideal voltage source, (b) an ideal current source, (c) an ideal open circuit, and (d) an ideal short, also known as ideal wire.

[2]We are using different symbols for ideal voltage sources and non-ideal voltage sources; some texts use the battery symbol for both.

Example 1.1

Mark the branch currents and the polarities in the following figure. Don't worry about what the "generic element" is for now. How are the branch currents related to each other? How are the voltages across the elements related to each other.



(a) **(b)**

Figure 1.4: An example to illustrate the passive sign convention.

Solution: We mark the figure in three steps.

1. The two voltage sources define their own polarities, so we mark them plus and minus according to the circuit symbol.
2. With that done, the passive sign convention dictates that the currents will be down through the two voltage sources, so we mark arrows and assign variables I_1 and I_2.
3. For the unknown element, we don't know either the polarity or the current direction, so we just have to guess. However, once we've chosen one, the passive sign convention chooses the other. So if we put voltage positive on the left (as we have), current flows to the right. We could just as well reverse both of them. We'll know that our guess is correct if we get positive results when we substitute numbers.

Now that the polarities and currents are marked, and necessary symbols are introduced, we can relate the values.

By KCL, the sum of all currents leaving a node must be zero, so for the top-left corner, both currents are leaving, so $I_1 + I_3 = 0$. For the top-right corner, I_3 is entering and I_2 is leaving, so $I_3 - I_2 = 0$. We could have set the sum of all currents leaving the node to zero, instead, which would have yielded $-I_3 + I_2 = 0$. These expressions are, of course, equivalent.

The voltages are related by KVL. If we start at the bottom left and go clockwise, summing the voltage *rises*, we get $+V_1 + (-V_3) + (-V_2) = 0$. If, instead, we sum the *drops*, we get $(-V_1) + V_3 + V_2 = 0$. Again, these are equivalent.

1.4.3 Resistance, Resistors, and Ohm's Law

When an electric current flows through a very good conductor, such as a copper wire, the electrons move very easily and very little work is required to move them. A copper wire has very low, almost zero resistance.

To force an electric current to flow through a very poor conductor, such as a piece of glass or an open switch, a practically infinite amount of energy would be required. In purely practical terms, electrons do not move through pieces of glass or open switches, and there is no current flow. No current flow means that no charge is moved and no work is done. A piece of glass or an open switch has very high, almost infinite resistance.

When an electric current flows through a thin wire, such as the filament in a lamp, the electrons can be moved, but considerable work is required to make the electrons move. The filament of a lamp and other devices on which electricity does work are said to have finite resistance. A special class of such materials with finite resistance, called **resistors**, exhibit a linear relationship between the voltage across them—and hence the work that must be done to push charge through them—and the current through them:

$$v = Ri, \tag{1.10}$$

where R is the **resistance**. Eq. (1.10) is called **Ohm's law**, and the unit of resistance is called the ohm (symbol "Ω," the Greek capital letter omega):

$$\text{ohm} = \frac{\text{volt}}{\text{ampere}}.$$

The resistor's circuit symbol and $i - v$ characteristics are shown in Fig. 1.5.

The inverse of resistance is called **conductance** G:

$$G = \frac{1}{R}, \tag{1.11}$$

and the unit for conductance is the siemens, symbol "S," defined as 1/Ω or A/V.

Note

Ohm's law, Eq. (1.10), is not a fundamental law of physics. It is strictly empirical, and it cannot be derived. It does, however, apply to a great many materials. For all intents and purposes, a resistor is *defined* as an element which obeys Ohm's law.

Power in Resistive Circuits

It is important to know how much power is being dissipated in a resistor. Every resistor has a maximum power rating. If this power is exceeded, the resistor will get overheated and quite literally burn up. Whenever a resistor is put into service, a power calculation should be performed to make sure that the resistor will not burn out.

One way to make a power calculation is to use Eq. (1.8): multiply the current through the resistor by the voltage across it. This is valid, but if you know only one, voltage or

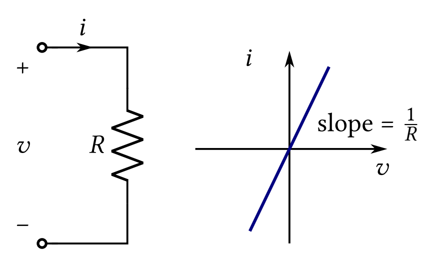


Figure 1.5: The circuit symbol for a resistor and its $i - v$ characteristics (at least in the US; In Europe and Asia, the resistor is usually indicated using what we call the "generic element" symbol shown in Fig. 1.2).

current, you must use Ohm's law to calculate the other. A couple of shortcuts can be derived by combining Eq. (1.8), with Eq. (1.10), Ohm's law. A direct substitution gives

$$p_\mathrm{a} = i(iR) = i^2 R. \tag{1.12}$$

Solving Ohm's law for current and substituting yields

$$p_\mathrm{a} = \frac{v}{R} v = \frac{v^2}{R}. \tag{1.13}$$

These equations should not be confused with the definition of power, but they are often very handy.

Example 1.2

For the circuit in Fig. 1.6(a), determine the power dissipated by the resistor. For the circuit in Fig. 1.6(b), determine the power supplied by the current source. Assume $V_\mathrm{s} = 1$ V, $I_\mathrm{s} = 10$ mA, and $R_\mathrm{L} = 1$ kΩ.

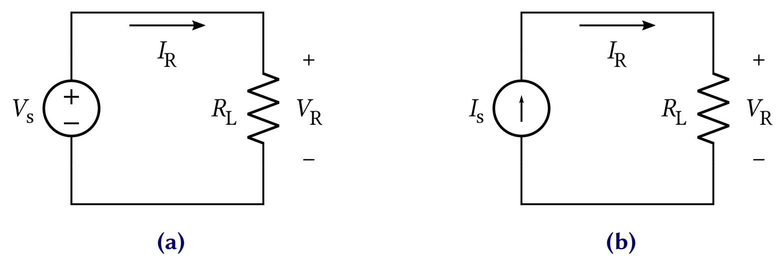


Figure 1.6: Resistors in combination with a voltage source and a current source.

Solution:

(a) From KVL, if we go clockwise starting in the bottom left, we have $+V_\mathrm{s} - V_\mathrm{R} = 0$. Therefore, $V_\mathrm{R} = V_\mathrm{s} = 1$ V. By Ohm's law, $I_\mathrm{R} = V_\mathrm{R}/R = (1\text{ V})/(1\text{ k}\Omega) = 1$ mA. We can calculate the power using Eq. (1.8): $P_\mathrm{a} = I_\mathrm{R} V_\mathrm{R} = (1\text{ mA})(1\text{ V}) = 1$ mW. Note that we *could* have skipped calculating the current and used Eq. (1.13): $P_\mathrm{a} = V_\mathrm{R}^2/R = (1\text{ V})^2/(1\text{ k}\Omega) = 1$ mW.

(b) The current source forces $I_\mathrm{s} = 100$ mA through it. Since there are no junctions, there is nowhere for this current to go except through the resistor, so $I_\mathrm{R} = I_\mathrm{s}$. Thus the voltage across the resistor is given by Ohm's law as (noting that, as drawn, I_R and V_R follow the passive sign convention) $V_\mathrm{R} = I_\mathrm{R} R = (10\text{ mA})(1\text{ k}\Omega) = 10$ V. From KVL, any path from the bottom to the circuit to the top must result in the same change in voltage, so if we increase 10 V going through the resistor, we must also rise 10 V going up through the current source: $V_{I_\mathrm{s}} = V_\mathrm{R}$. The power delivered by some element is given by Eq. (1.9) as $P_\mathrm{d} = -iv$. But, since the current is coming *out* of the positive terminal of the current source, we must enter either i or v as negative, so

$$P_\mathrm{d} = -(-I_\mathrm{s})(V_{I_\mathrm{s}}) = -(I_\mathrm{s})(-V_{I_\mathrm{s}}) = (10\text{ mA})(10\text{ V}) = 100\text{ mW}.$$

Table 1.1: Resistivities of some common materials at 293 K or 20 °C[1].

Material	ρ (10^{-8} Ωm)
Silver	1.59
Copper	1.68
Gold	2.21
Aluminum	2.65
Nickel	6.93

Resistivity and Conductivity

Resistance can be qualitatively described as an element's resistance to the flow of charge. It is a property not only of the material of which the element is made but also of the geometry of the element.

You can compare it to a tube's resistance to the flow of water. The narrower you make the tube, the more effort it will take to get a certain amount of water to flow through it in a given amount of time. Likewise, the *longer* the tube is, the more difficult it is to get a given amount of water through.

Resistors are similar. If your resistor has a uniform cross-section, its resistance can be written as

$$R = \rho\frac{\ell}{A}, \tag{1.14}$$

where ℓ is the length of the resistor, A is its cross-sectional area, and ρ (Greek letter rho) is the **resistivity** of the material of which the resistor is made. Resistivity is simply a property of the material, and it can be found in tables in most textbooks (see Table 1.1) and on the internet.

Conductivity σ is simply the inverse of resistivity:

$$\sigma = \frac{1}{\rho}. \tag{1.15}$$

1.4.4 Capacitance and Capacitors

A capacitor is nothing more than an open circuit, yet it is the most useful open circuit ever discovered. A **capacitor** consists of two conductors separated by an insulator (which may be vacuum). it permits electric charge to be stored and released in a controllable and repeatable manner. This stored charge has opposite signs on the two conductors, establishing an electric field between them, and this field stores energy. The circuit symbol for a capacitor is shown in Fig. 1.7.

Traditionally, the term capacitor is used only for those devices such that the magnitude of the charge stored on *each* conductor, q, varies linearly with the voltage v between the conductors, and the constant of proportionality is called the **capacitance** C:

$$C \equiv \frac{q}{v}. \tag{1.16}$$

While this definition may seem to suggest that capacitance depends on the charge on the conductors and the voltage between them, *this is not the case.* Capacitance is determined by the design of the device, and the charge and voltage vary together in such a way that their ratio is a constant.

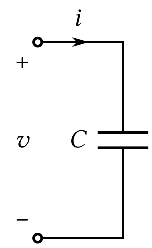


Figure 1.7: The circuit symbol for a capacitor.

Capacitance is measured in farads (F), and typical capacitors have capacitances of a few pF to a few hundreds of μF. Larger capacitance (even much larger) is possible and has many applications, but it is uncommon in typical lab- or homemade circuits.

We can solve Eq. (1.16) for q and take the time derivative:

$$\frac{dq}{dt} = C\frac{dv}{dt}.$$

But the rate of change of q is current, so

$$i = C\frac{dv}{dt}. \tag{1.17}$$

Note that this current does not pass *through* the capacitor, since the two conductors are separated by an insulator. However, since charge cannot build up on any element in the lumped matter discipline, if a small amount of charge enters the positive conductor, the same positive charge leaves the negative conductor, making it more negative. In other words the charge building up on or coming off the capacitor establishes this current i *around* the capacitor. But, since the same current that goes into one terminal goes out of the other terminal, it looks *as if* the current passes through the capacitor, and we will often speak as if it did.

Example 1.3

If the current in a circuit near a capacitor is 120 mA and the rate of change of voltage is 1.5 V/ms, what is the capacitance?

Solution: Solving Eq. (1.17) for C gives

$$C = \frac{i}{dv/dt} = \frac{120 \times 10^{-3}\ \text{A}}{1.5 \times 10^{3}\ \text{V/s}} = 8.0 \times 10^{-5}\ \text{F} = 80\ \mu\text{F}.$$

Eq. (1.17) tells us that we cannot change the voltage across a capacitor in zero time. If we tried, the current would be infinite. This equation also tells us that if we maintain a constant current through a capacitor, the voltage across it will change linearly with time. This is the basis of the linear sweep in an oscilloscope.

Energy Storage in Capacitors

As we mentioned before, capacitors store energy in their electric fields. The change in energy stored by a capacitor equals the work done on it, so we can rearrange Eq. (1.4) and

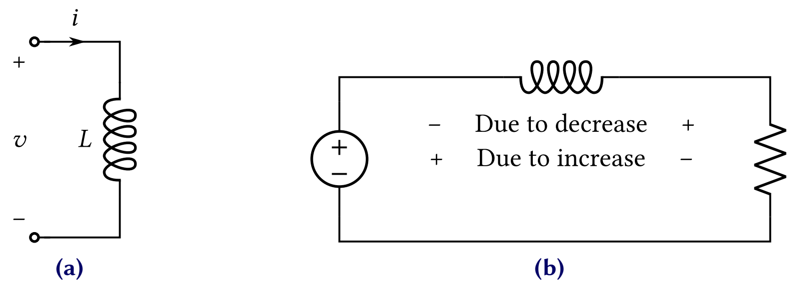


Figure 1.8: The circuit symbol for an inductor, and an illustration to help explain the signs.

substitute Eqs. (1.5) and (1.17) to get

$$dw = p\,dt = iv\,dt = Cv\frac{dv}{dt}\,dt = Cv\,dv.$$

Integrate to get the energy stored by the capacitor:

$$U_C = \int dw = C\int v\,dv = \frac{1}{2}Cv^2 + U_0.$$

Make the natural assumption that if $v = 0$, there is no electric field, and define that as the zero-point of potential energy, so

$$U_C = \frac{1}{2}Cv^2. \tag{1.18}$$

1.4.5 Inductance and Inductors

A solenoid (coil of wire) carrying a current i produces a magnetic field

$$B = \frac{\mu_0 N i}{\ell},$$

where μ_0 is the permeability of free space, N is the number of windings of the coil, and ℓ is the length of the solenoid. The magnetic flux of this solenoid is $\Phi_B = BA$, where A is the cross-sectional area of the solenoid, so

$$\Phi_B = \frac{\mu_0 N^2 i}{\ell}A,$$

where N is squared because the effective area of the solenoid is N times the area of each circular loop. By Faraday's law, if the current through the solenoid starts to change, it will induce an emf to oppose this change given by

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 N^2 A}{\ell}\frac{di}{dt}. \tag{1.19}$$

Such a coil of wire is the prototypical example of an **inductor**. An inductor is an element which resist changes in current and stores energy in magnetic fields. The current through and the voltage across an inductor are related by

$$v = L\frac{di}{dt}, \tag{1.20}$$

where L is the **inductance** of the inductor. It's definition is

$$L = \frac{\Phi_B}{i}, \tag{1.21}$$

which, for a solenoid, is

$$L = \frac{\mu_0 N^2 A}{\ell}. \tag{1.22}$$

The circuit symbol for the inductor is shown in Fig. 1.8(a), though the current direction relative to the polarity may be a little confusing. Fig. 1.8(b) may help somewhat. If the battery voltage begins to decrase, the current i coming tends to decrease, then $di/dt < 0$. The inductor wants to push current upward (or leftward) through it to counteract this, so $v < 0$, as we'd expect. On the other hand, if the voltage source increases its voltage, the current would tend to increase. The inductor wants to prevent that, and start pushing current upward (or leftward) through it. So now $di/dt > 0$, and $v > 0$

Note that while the inductor opposes change in current, it only has a finite amount of energy stored in its magnetic field It cannot oppose the battery indefinitely. The inductor cannot *prevent* the change in current, but it can slow it down for some time.

Energy Storage in Inductors

Just as a capacitor can store and release energy, so can an inductor. Whereas the capacitor's energy was stored in the electric field between the conductors, the inductor's is in the magnetic field produced by the current flowing through the coil of wire.

Return to Eq. (1.4) and rearrange, substituting Eqs. (1.5) and (1.20), to find

$$dw = p\,dt = iv\,dt = Li\frac{di}{dt}\,dt = Li\,di.$$

Integrate both sides to get

$$U_\mathrm{L} = L\int i\,di = \frac{1}{2}Li^2 + U_0.$$

Choose the most natural convention that potential energy is zero if no current flows, so

$$U_\mathrm{L} = \frac{1}{2}Li^2. \tag{1.23}$$

Example 1.4

A relay coil has an inductance of 95 mH. A transistor switch turns off a current of 40 mA through the coil in a time of 1 μs. What is the magnitude of the voltage spike that is produced?

Solution:

$$v = L\frac{di}{dt} \approx L\frac{\Delta i}{\Delta t} = 95\ \mathrm{mH}\frac{(40\ \mathrm{mA})}{(1\ \mu\mathrm{s})} = 3800\ \mathrm{V}$$

References

[1] J. R. Rumble, ed., *CRC Handbook of Chemistry and Physics*, 99 (Intern (CRC Press/Taylor & Francis, Boca Raton, FL, 2018).

[2] D. L. Eggleston, *Basic Electronics for Scientists and Engineers*, 1st ed. (Cambridge University Press, Cambridge, UK, 2011).

Suggestions for Further Reading

D. L. Eggleston, *Basic Electronics for Scientists and Engineers*, 1st ed. (Cambridge University Press, Cambridge, UK, 2011)

Exercises and Problems

1. Two spheres are charged with +25 μC and +50 μC respectively. The distance between them is 20 m. What is the force exerted between the two spheres? Is the force attractive or repulsive?
2. Express the ampere in terms of kilograms, meters, and seconds.
3. If a current of 500 mA flows for 11 minutes, how much charge is moved?
4. If a charge of 1026 C was transferred at a constant rate in a time of 3 hours, what was the current?
5. A current of 25 mA must be left on until a charge of 10 000 C has been moved. How long must the current be left on? Express your answer in days, hours, minutes and seconds.
6. Express the volt in terms of kilograms, meters, and seconds.
7. If 300 J of work are done on 25 C of charge, what was the potential through which the charge was moved?
8. If 12 C of charge are moved through a potential of 120 V, how much work was done?
9. If 420 J of work are done while moving a certain amount of charge through a potential difference of 28 V, how much charge was moved?
10. If 15 A is drawn from a 12 V battery, what is the power?
11. What is the current drawn by a 100 W, 120 V light bulb?
12. For proper operation, an electroplating cell requires the movement of 1.26×10^5 C per hour. The cell voltage is 3.3 V. How much power is required to keep the cell in continuous operation?
13. Express the ohm in terms of kilograms, meters, and seconds.
14. A current of 55 mA is flowing through a 270 Ω resistor. What is the voltage drop across the resistor?
15. When a potential of 12 V is applied to an unknown resistor, a current of 24.74 mA flows. What is the resistance of the resistor?
16. If a 56 Ω resistor is placed across a 15 V power supply, how much current will flow?
17. A resistor substitution box contains the following resistor values:

    - First decade: 15, 22, 33, 47, 68 and 100 Ω.
    - Second decade: 150, 220, 330, 470, 680 and 1000 Ω.

    Each successive decade follows the same pattern. The highest resistance is 10 MΩ. Each resistor can dissipate a maximum of 1 W without burning out.

    (a) What is the minimum resistance setting which can be safely connected across the 120 V power line?
    (b) What is the minimum resistance setting which can be safely connected across a 20 V power supply?
    (c) What is the minimum resistance setting which can be safely connected across a 12 V power supply?
    (d) What is the minimum resistance setting which can be safely connected across a 5 V power supply?

18. How much current can safely be run through

(a) a 39 Ω, 1/2 W resistor„
(b) a 470 Ω, 1/4 W resistor,
(c) a 27 Ω, 2 W resistor, and
(d) a 560 Ω, 1 W resistor?

19. How much charge is stored on a 7000 μF capacitor when it is charged to a voltage of 18 V?
20. If a 2000 μF capacitor is being discharged with a current of 1 A, how much will its voltage change in 8 ms?
21. A circuit has a capacitance of 100 pF. How much current must the circuit deliver to the capacitor in order to change the voltage at a rate of 13 V/μs?

Chapter 2

Methods of Circuit Analysis

In this chapter, we will explore various methods for solving circuits. The first, and simplest, is learning how various elements combine in series and in parallel. Even with such combinations, KVL and KCL can remain unwieldy for more complicated circuits. We will see two very powerful methods which follow from KVL and KCL, but enable us to significantly reduce the number of equations we need to solve, called the *node voltage method* and the *loop current method*.

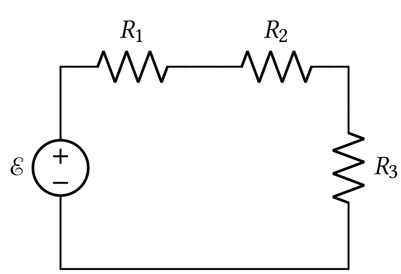
2.1 Applying KVL and KCL Directly

Electric circuits get much more complex than the flashlight shown back in Fig. 1.1. We will work on a few slightly more complicated circuits by applying KVL and KCL. We will then examine how the same results may be obtained once we learn to combine elements in series and parallel.

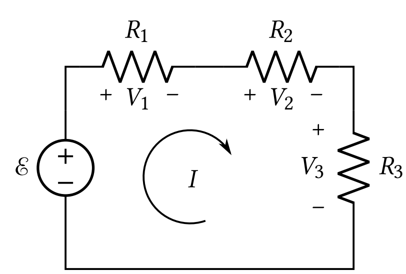
2.1.1 A Series Circuit

A circuit is in **series** if all of its elements are arranged so that there is only *one* path that can be followed from a given point, around the circuit, and back to the same point.

A simple example of a series circuit is Fig. 2.1(a). The letter $\mathcal{E}$ is used to represent the emf or potential difference across the voltage source, which is the only element which can *cause* current, so we can assume that the current will flow clockwise out of the positive terminal of the source, through the resistors, and back into the negative terminal. Thus, we indicate the direction of our current I as clockwise in Fig. 2.1(b). Once we've marked the direction of the current, we specify the voltage drops across the resistors. The symbol



(a)



(b)

Figure 2.1: Schematic diagram of a simple series circuit containing a voltage source and three resistors.

V_1 is used for the potential difference across the resistor R_1. In general, the symbol V_n will be used for the potential across any resistor R_n.

Once the direction of the current is chosen, the signs of the voltages across the resistors is determined by the passive sign convention—the positive terminal of each resistor is the terminal where current *enters*. Thus, we give that side of each resistor a + sign, and we give the other side a − sign.

Current in Series Circuits

Let us use the general symbol I_n for the current in resistor R_n and the letter I for the current through the battery.

Remember that current could be expressed as the number of electrons per second passing a point in the circuit. As electrons move around the circuit, there are no junctions, or "side roads," for them to take. Electrons cannot pile up at any particular place in the circuit. If there were a pile-up, the concentration of negative charge would repel additional electrons from that vicinity, and the pile-up would soon be gone. Thus, pile-ups do not happen. Therefore, the number of electrons per second passing any point in the circuit of Fig. 2.1 is the same as the number of electrons per second passing any *other* point in the circuit. In other words, the current at any point in the circuit is exactly the same as the current at any other point in the circuit. This can be stated in equation form as

$$I = I_1 = I_2 = I_3. \tag{2.1}$$

This is merely an application of Kirchhoff's current law, Eq. (1.6).

Voltage in Series Circuits

Remember that

1. potential difference, or voltage, is work per unit charge;
2. the work done on an object equals its change in potential energy;
3. energy is conserved;
4. charge is conserved;
5. the current in any resistor in Fig. 2.1 is the same as the current in any other resistor and the same as the current in the battery.

It follows from (2) and (3) that the amount of work done *by* the battery is equal to the sum of the work done *on* each resistor:

$$W_{\mathcal{E}} = W_1 + W_2 + W_3.$$

Consider Eq. (1.1), $v = w/q$. Rearrange, and use capital letters to conform to our problem (everything is this problem is constant in time, so, by convention[1] we use capital letters). Then

$$Q\mathcal{E} = QV_1 + QV_2 + QV_3.$$

It follows from (4) and (5) that all of the Qs are the same (otherwise we'd have used different symbols or subscripts, of course), so charge Q cancels, leaving

$$\mathcal{E} = V_1 + V_2 + V_3 \qquad \text{or} \qquad -\mathcal{E} + V_1 + V_2 + V_3 = 0. \tag{2.2}$$

This is just an example of KVL [Eq. (1.7)], which states that the algebraic sum of all voltage drops around any closed loop is equal to zero.

[1]It is a common convention, but not a universal one. In this text, we will follow it for the most part but not with perfect consistency.

Finding the Current By Eq. (2.2), Ohm's law, and Eq. (2.1),

$$\mathcal{E} = I_1R_1 + I_2R_2 + I_3R_3 = IR_1 + IR_2 + IR_3 = I(R_1 + R_2 + R_3),$$

so the current is

$$I = \frac{\mathcal{E}}{R_1 + R_2 + R_3}. \tag{2.3}$$

Reviewing Voltage Rises and Drops Let us review how to determine signs when writing an KVL equation. If you understand how all of the signs worked out above, feel free to skip ahead to the Example below, and remember that you can return to this section later if you find yourself confused (the signs tend to be confusing to beginners, so it's worth being a little repetitive on the topic).

All voltages are really *potential differences*. A potential difference or voltage difference can be either a voltage *rise* or a voltage *drop*. Although it may be stating the obvious, a rise is an increase in potential or voltage, and a drop or fall is a decrease in potential or voltage. If we start at a particular point and move around the circuit, we may move from a point of low potential to a point of higher potential. We have traversed a potential rise. If we move from a point of high potential to a point of higher potential we have again traversed a potential rise. The starting point makes no difference; it is the *change* that is important. Similarly if we move from a point of any given potential to a point of lower potential we have traversed a potential drop or fall.

For example, if we traverse the voltage source in Fig. 2.1 from bottom to top, we have traversed a rise. Conversely, if we go from top to bottom, we have traversed a drop. From this, you can see that a battery is not automatically a potential rise; it depends on the direction of travel.

It is easy to tell the direction of the potential or polarity of a voltage source because of the polarity markings. A voltage source will always have the same polarity (positive in the symbol) regardless of whether it is being charged (current flowing into the positive terminal) or discharged (current flowing out of the positive terminal). The positive terminal of the battery is always positive, no matter which way the current is flowing.

Resistors and other passive devices are another story. Conventional current always flows out of the negative end of a passive device. Thus, the end where the current goes in is more positive than the end where the current comes out.

In solving Kirchhoff's equations, it is very important to give the proper sign to each quantity. If an incorrect sign is given to a particular value, the result will be incorrect. Here are some rules to follow in writing Kirchhoff's equations.

1. Note the polarity on all voltage sources. If your voltage source is ideal, it is already indicated. If it is a battery, place a minus (−) sign next to the negative terminal (short line) and a plus (+) sign next to the positive terminal (long line).
2. Assume a direction for the current. Making all current directions clockwise is a typical, though not necessary, assumption. Don't be concerned if you can't tell which direction the current is *actually* flowing. The numeric answer will have a sign which will tell us whether the initial assumption was right or wrong.
3. Mark the polarity on all passive devices (resistors) by placing a plus (+) sign at the end where the current enters and a minus (−) sign at the end where the current exits.

Now that you have polarity markings on each part in the circuit, you are almost ready to write a KVL equation.

Do not be concerned about the fact that there are + and − signs next to each other on opposite ends of the same wire. The signs refer to that particular circuit element alone. The positive end is more positive than the negative, and the negative end is more negative than the positive end. The signs say nothing about what the potential is with respect to other parts of the circuit.

KVL states that the sum of all voltage drops is equal to zero or that the sum of all voltage rises is equal to zero. The only difference between the two resulting equations will be that one has been multiplied through by −1 relative to the other. If you are summing the rises, a drop is a negative rise; conversely a rise is the negative of a drop. Therefore, if we sum the drops, a drop is a positive quantity and a rise is a negative quantity.

Apply the three rules above to Fig. 2.1(a), starting at the lower left corner. The battery $\mathcal{E}$ is a rise, which is a negative drop. Thus, we write the emef $\mathcal{E}$ with a minus sign in Eq. (2.2). The drops across the resistors are just that and are entered as positive quantities.

Example 2.1

Write a KVL equation for the circuit of Fig. 2.2.

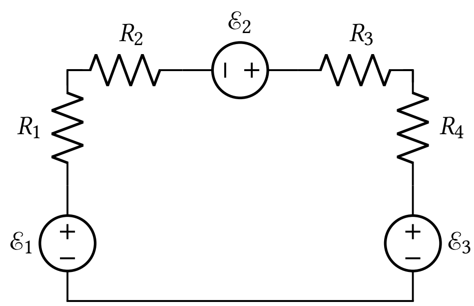


Figure 2.2: Series circuit with multiple voltage sources.

Solution: First, the the voltage sources are ideal, so their polarities are already marked. Second, choose a current direction. Here we choose clockwise arbitrarily; indicate it by drawing an arrow on the figure. Put + and − signs on each resistor as follows. The lower end of R_1 is positive and the upper end is negative, since, by our clockwise-current assumption, the current enters from the bottom. The left ends of R_2 and R_3 are positive, and the upper end of R_4 is positive. Of course, the other end of each resistor is negative. Now we can write the equation. Start at the lower left corner and move around the circuit in a clockwise direction. If we sum the drops we get

$$-\mathcal{E}_1 + V_1 + V_2 - \mathcal{E}_2 + V_3 + V_4 + \mathcal{E}_3 = 0.$$

On the other hand, if we sum the voltage rises we get

$$\mathcal{E}_1 - V_1 - V_2 + \mathcal{E}_2 - V_3 - V_4 - \mathcal{E}_3 = 0.$$

The second equation is just the first equation multiplied by −1. If we were to plug in numbers and solve them, both would give the same answer.

2.1.2 A Parallel Circuit

There is no clear, broadly accepted definition of a parallel circuit (parallel *circuit elements* is another matter, as we shall see in the next section). For our purposes just here, we'll define a parallel circuit as one which has two or more independent[2] paths which can be followed from a given point in the circuit back to the same point. A simple parallel circuit is shown in Fig. 2.3.

Voltage in Parallel Circuits

The alternative statement of KVL [Eq. (1.7)] is that the potential difference between any two points does not depend on the path we take between those two points. If we start at the bottom of Fig. 2.3 and move to the top (since the bottom is connected only by ideal wire, which drops no potential, and the top is also connected by only ideal wire), we must get the same potential difference no matter what branch we go through.

If we go through the battery, we rise by a voltage $\mathcal{E}$. Alternatively, if we go up through one of the resistors, we rise a voltage $+V_1$, $+V_2$, or $+V_3$. Since our starting points and ending points are the same, KVL tells us that

$$\mathcal{E} = V_1 = V_2 = V_3.$$

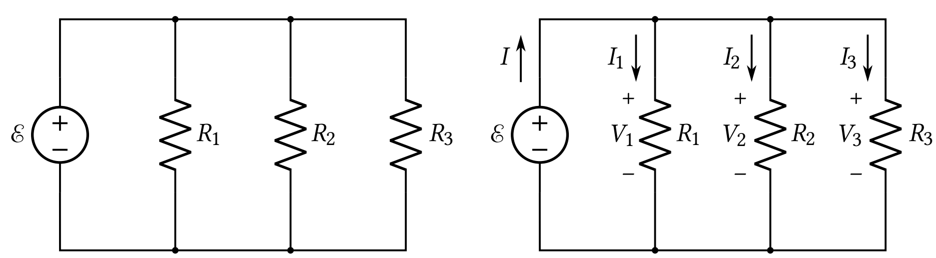
In a (simple) parallel circuit the voltage across any one element is the same as the voltage across any other element in the same circuit.

Current in Parallel Circuits

Fig. 2.4 is the map of a rather strange system of roads. The system is one-way, left to right. Cars are fed into the system at the left at a rate measured in cars/hour. At the first junction some cars go left, some go straight and some go right. The cars go through the three parallel roads and join up again at the rejoin point. Cars cannot get out of the system except at the ends, and no cars from the outside can get in. Cars exit on the right at the same rate as they enter on the left. No stopping.

When the car-current (cars/hour) splits, it is obvious that the sum of the car-currents (K_n) in the three paths is equal to the car-current (K) in the main path.

$$K = K_1 + K_2 + K_3$$



(a) **(b)**

Figure 2.3: A schematic diagram of a simple parallel circuit.

[2]The precise definition of "independent" is the point of contention. It's not going to matter text, or in life in general.

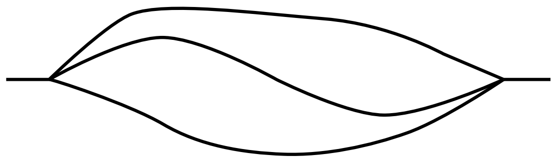


Figure 2.4: Map of a strange road system.

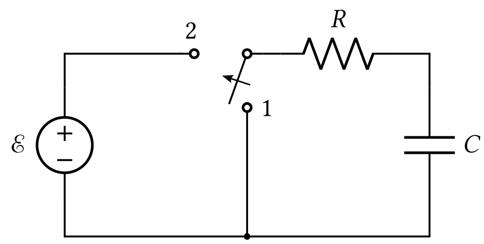


Figure 2.5: A circuit for charging and discharging a capacitor.

Look at Figure Fig. 2.3, and visualize the electrons in the circuit doing the same as the cars on the road. This is the justification for writing the equation

$$I = I_1 + I_2 + I_3,$$

which is a special case of Kirchhoff's current law [Eq. (1.6)]. Kirchhoff's current law says that the sum of all currents flowing into our out of a node is equal to zero. If current going in is defined as positive, then current coming out is negative. In Fig. 2.3, the top line is one node, and the bottom line is another. If we sum the currents *entering* the top line, we find

$$I - I_1 - I_2 - I_3 = 0. \tag{2.4}$$

If we sum the currents *leaving* the top line, we have

$$-I + I_1 + I_2 + I_3 = 0. \tag{2.5}$$

The only difference between Eq. (2.4) and Eq. (2.5) is that one is the negative of the other. Numerical solution of both equations will give the same answer. In this problem, there are only two nodes. Therefore, for the bottom line, the currents *leaving* sum to Eq. (2.5), and the currents *entering* sum to Eq. (2.4)—exactly the reverse of the sums at the top line.

2.1.3 The Series *RC* Circuit

Consider the circuit in Fig. 2.5. Suppose the switch starts out in position 1 to ensure that there is no initial charge on the capacitor. Then the switch is changed to position 2, and the capacitor begins to charge. The bottom plate will receive a negative charge while the top plate will receive a positive charge. Kirchhoff's voltage equation for this circuit at any instant of time is

$$\mathcal{E} = iR + \frac{q}{C}. \tag{2.6}$$

The battery voltage $\mathcal{E}$, the resistance R, and the capacitance C do not vary with time, while the current i and the charge on the capacitor, q, do. Substitute $i = dq/dt$ and rearrange to get

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{\mathcal{E}}{R}.$$

We can solve this by finding a particular solution, solving the homogeneous equation, adding the two solutions, and applying the initial conditions. We complete the first two steps by guessing-and-checking, and, fortunately, the simplest guess for the particular solution is just a constant (call it q_{p}). The time-derivative in the first term will make the constant disappear, leaving us with $q_{\mathrm{p}}/(RC) = \mathcal{E}/R$, or

$$q_{\mathrm{p}} = \mathcal{E}C. \tag{2.7}$$

The homogeneous solution is the solution to the equation with the driving-term set to zero:

$$\frac{dq_{\mathrm{h}}}{dt} + \frac{1}{RC}q_{\mathrm{h}} = 0.$$

Guess a solution of the form

$$q_{\mathrm{h}}(t) = Ae^{-t/\tau}. \tag{2.8}$$

Substitution yields

$$-\frac{A}{\tau}e^{-t/\tau} + \frac{A}{RC}e^{-t/\tau} = 0.$$

This can hold for all t if and only if

$$\tau = RC. \tag{2.9}$$

The quantity τ (lowercase Greek letter "tau") is called the **time constant** of the circuit. Now, to find the general solution, add together the particular solution Eq. (2.7) and the homogeneous solution Eq. (2.8) to find

$$q(t) = \mathcal{E}C + Ae^{-t/\tau}.$$

By Eq. (1.16), the voltage across the capacitor is then

$$v(t) = \frac{q}{C} = \mathcal{E} + \frac{A}{C}e^{-t/\tau} \tag{2.10}$$

Charging the Capacitor As the capacitor begins to charge, the charge is zero by our original assumptions, and, therefore, so is the voltage across it. Thus, at $t = 0$, Eq. (2.10) is

$$v(0) = \mathcal{E} + \frac{A}{C}e^{-0/\tau} = \mathcal{E} + \frac{A}{C} = 0,$$

so $A/C = -\mathcal{E}$, and

$$v(t) = \mathcal{E}\left(1 - e^{-t/\tau}\right). \tag{2.11}$$

The current through[3] the capacitor is then, by Eq. (1.17),

$$i = C\frac{dv}{dt} = C\left(-\frac{1}{\tau}\right)\left(-\mathcal{E}e^{-t/\tau}\right) = \frac{\mathcal{E}}{R}e^{-t/\tau}.$$

[3] As we mentioned when we introduced the capacitor, no charge *actually* throws *through* it. When a capacitor is being charged, electrons flow off of the positive therminal and onto the negative terminal in such a way that it *looks* like charge flows through it, and it is usually easier to speak as if the charge is flowing through it.

As time elapses, the voltage will quickly approach $\mathcal{E}$ as the exponential term decays (how quickly will depend on the value of $\tau = RC$), while the current will quickly fall to zero. In other words, *at long time scales, a capacitor behaves like an open circuit.*

To quantify this somewhat, we remark that the voltage will reach approximately $1/e = 0.368$, or 36.8 %, of its final value after one time constant τ. After five time constants, it will be at $(1 - e^{-5}) = 0.993$, or 99.3 %, of its maximum value. Current, on the other hand, is decreasing: it falls to 36.8 % of its initial value of $\mathcal{E}/R$ in one time constant and drops to a miniscule 0.67 % after five. For typical element values of $R = 1\ \mathrm{k\Omega}$ and $C = 1\ \mathrm{\mu F}$, five time constants is only $5\tau = 5RC = 5$ ms.

Discharging the Capacitor If we charge the capacitor long enough for it to reach its final value of $\mathcal{E}$ and then flip the switch in Fig. 2.5 back to position 1, so that the capacitor can discharge through the resistor, what happens?

All of the equations from Eq. (2.6) through Eq. (2.10) require only one simple change: set $\mathcal{E} = 0$. Now the *initial* voltage across the capacitor is $v(0) = \mathcal{E}$. So we can apply these two modifications to find

$$v(0) = (0) + \frac{A}{C}e^{-(0)/\tau} = \mathcal{E},$$

so $A/C = \mathcal{E}$, and

$$v(t) = \mathcal{E}e^{-t/\tau}.$$

The current in then just

$$i = C\frac{dv}{dt} = -\frac{C\mathcal{E}}{\tau}e^{-t/\tau} = -\frac{\mathcal{E}}{R}e^{-t/\tau}.$$

General Remarks on the *RC* Circuit We will summarize and restate some qualitative remarks about the behavior of capacitors in circuits.

- Initially, a capacitor acts like a short circuit. The voltage drop across it is zero, while the current is determined by the rest of the circuit.
- After a long time (though, due to the exponential dependence and small values of τ, what constitutes "long" is generally only a few milliseconds), a capacitor acts like an open circuit: the current through it is zero, while the voltage across it is determined by the rest of the circuit.
- The *voltage across a capacitor changes continuously*. The only way to get sudden jumps in the voltage is with an infinite voltage pulse (a "delta function" pulse). Such pulses may be useful in analysis, but they are only approximations.
- The current through a capacitor, on the other hand, may undergo large jumps in small (theoretically zero) time.

2.1.4 The Series *RL* Circuit

Fig. 2.6 is a schematic of a resistive-inductive (*RL*) circuit. The switch is initially placed in position 1 to ensure that there is no current in the circuit. When the switch is changed to position 2, the source's potential is applied to the series resistor and inductor circuit. For an infinitesimal instant of time, no current will flow in the circuit: the inductor is creating an emf that opposes the voltage source. If the voltage across the source and the inductor have the same magnitude and sign, Kirchhoff's law says that there is no voltage drop across the resistor. If there is no voltage across the resistor, there is no current through it. Because this is a series circuit, no current through the resistor means no current flows

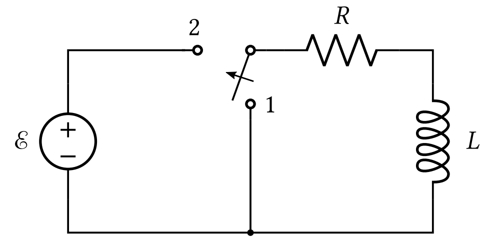


Figure 2.6: A series *RL* circuit with a switch.

anywhere in the circuit. But this cannot last for a finite length of time: the voltage source overpowers the battery, and current begins to flow, approaching $\mathcal{E}/R$ (assuming that the inductor has zero resistance, which is a a highly invalid assumption).

Now let us write KVL for the circuit of Fig. 2.6:

$$\mathcal{E} = iR + L\frac{di}{dt}, \qquad \text{or}$$

$$\frac{di}{dt} + \frac{i}{L/R}i = \frac{\mathcal{E}}{L}. \tag{2.12}$$

This has exactly the same form as the differential equation for charge on the capacitor in the series *RC* circuit, and the solution is

$$i(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right), \tag{2.13}$$

where the *RL* time constant is

$$\tau \equiv \frac{L}{R}. \tag{2.14}$$

This equation tells us that the current starts out as if the inductor were an open circuit, $i(0^+) = 0$. The current rises to exponentially approach $\mathcal{E}/R$.

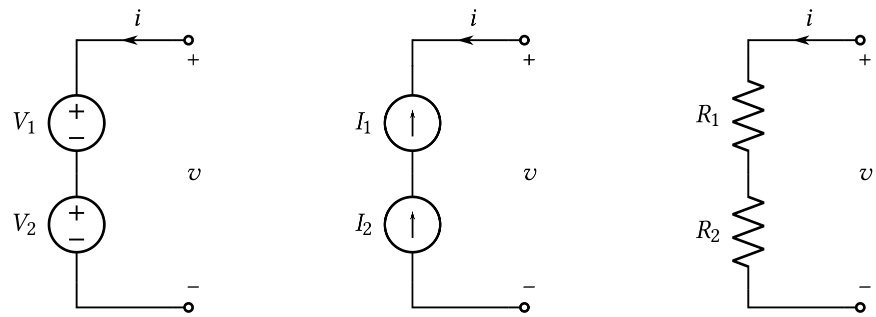
It is left as an exercise to the reader to show that he voltage across the inductor is

$$v_{\text{L}} = \mathcal{E}e^{-t/(L/R)}. \tag{2.15}$$

With the capacitor, current was decreasing, and after five time constants, it was practically zero. Now, the current is *increasing*, and after five time constants the current will be 99.3 % of $\mathcal{E}/R$, the value it would have if the inductor were absent.

General Remarks on the *LC* Circuit Inductors, in many ways, are "inverses" of capacitors, so they are sometimes called **dual** to capacitors. Compare the following points to the general remarks on the RC circuit given above.

- Initially, an inductor acts like an open circuit—there is a voltage spike, but no current flows.
- After a long time, an (ideal) inductor acts like a short circuit: the voltage across it is zero, while the current through it is determined by the rest of the circuit.
- The *current through an inductor changes continuously*. The only way to get sudden jumps in the current is with an infinite current pulse.
- The voltage across an inductor may undergo sudden jumps.



(a) **(b)** **(c)**

Figure 2.7: Voltage sources, current sources, and resistors in series.

The only significant change between this recap and the one for capacitors is that "voltage across" and "current through" have been swapped. Keep this in mind, and you will have less that you need to explicitly memorize.

Real Inductors In the foregoing, inductance and resistance were treated as if they were separate components of the circuit. In practice, inductors always have resistance. Inductors are coils of wire, and wire has resistance. It is impossible to make an inductor which does not have any resistance (at least not without superconductors, and all of those have to be kept too cold to be relevant in everyday electronics).

For purposes of circuit analysis, the inductance of a coil and its resistance are shown schematically as separate circuit elements. In the circuit of Fig. 2.6, it is possible that the resistor and inductor which are shown actually represent a single coil of wire.

2.2 Combining Elements in Series and Parallel

2.2.1 Series Combinations

Two elements are said to be in **series** if there are no junctions between them. In other words, *all* of the current flowing into the first element flows out of it and into the second. Let's explore how this works for the most basic circuit elements.

Voltage Sources in Series

Consider two voltage sources in series, as shown in Fig. 2.7(a). Recall that an ideal voltage source always maintains its specified voltage across its terminals. If we start at the − terminal and go clockwise through the two sources, we rise $+V_2$ and then $+V_1$ to give a total change of $+(V_1 + V_2)$. On the other hand, if we go counterclockwise, we rise a voltage $+v$. By KVL, any path from one point to another must result in the same change in potential, so

$$v = V_1 + V_2.$$

This can be generalized to any number of voltage sources in series, so if $V_1, V_2, \ldots, V_N$ are all in series, the sources can all be combined into a single ideal voltage source rated at

$$V_{\text{eq}} = V_1 + V_2 + \cdots + V_N = \sum_{n=1}^{N} V_n. \tag{2.16}$$

A simple, common (but non-ideal) application is putting two 1.5 V batteries in series to get 3.0 V at the output.

Current Sources in Series

Two current sources are shown in series in Fig. 2.7(b). Recall that the job of a current source is to ensure that the current through it is its rated value. If $I_1 \neq I_2$, this is impossible in this arrangement—the sources will end up fighting each other. Unless you know what you are doing, don't put two current sources in series.

Resistors in Series

Resistors in series, as shown in Fig. 2.7(c), is the most common of the three situations we are considering here.

As we go clockwise from the − terminal toward the top, we first reach resistor R_2. Note that we're going opposite to the direction of current, so the potential increases as we go across this resistor. The current through the resistor is i, so, by Ohm's law, the increase in voltage is given by iR_2. We then go through R_1, again increasing in voltage, according to Ohm's law, by iR_1. Thus, the total increase in voltage as we move clockwise is $\Delta v = iR_2 + iR_1 = i(R_1 + R_2)$.

Meanwhile, if we go counterclockwise, we rise by voltage $\Delta v = v$. By KVL, the two expressions must be equal, so

$$v = i(R_1 + R_2).$$

If we compare this expression with Ohm's law, these two resistors behave as a single resistor with resistance

$$R_{\text{eq}} = R_1 + R_2.$$

As with voltages, this can be generalized to as many resistors as you like. If you have N resistors in series, they can be combined into a single resistor of resistance

$$R_{\text{eq}} = R_1 + R_2 + \cdots + R_N = \sum_{n=1}^{N} R_n. \tag{2.17}$$

Logical Checks In any problem-solving course it is a good idea to have in mind some rules of logic for checking answers. You should look at an answer and ask yourself, "Does that make sense?" From time to time, this book will give some common sense rules under the heading of "Logical Checks."

When finding the equivalent of series resistors, the equivalent will always be greater than the largest single resistor.

Capacitors in Series

Now consider capacitors in series, as shown in Fig. 2.8(a). Starting at the bottom and going counterclockwise, we rise voltage v, while, if we go clockwise, we rise $v_1 + v_2$. Thus, by KVL,

$$v = v_1 + v_2.$$

By Eq. (1.16), $q = Cv$, so

$$\frac{q}{C_{\text{eq}}} = \frac{q_1}{C_1} + \frac{q_2}{C_2}. \tag{2.18}$$

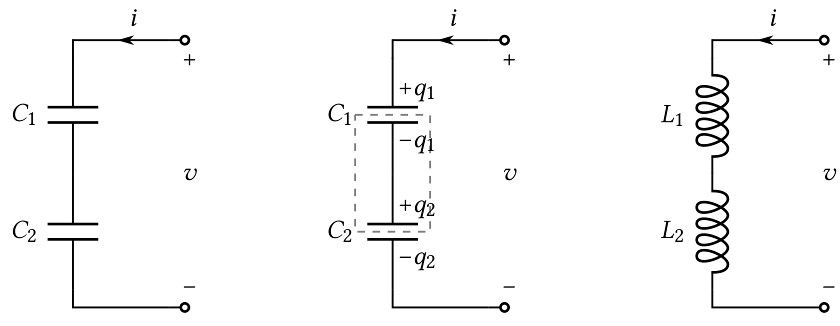


Figure 2.8: (a) Two capacitors in series. (b) The same two capacitors, with some quantities labeled. (c) Two inductors in series.

To see how the charges are related, look at Fig. 2.8(b). By the nature of capacitors, the charges on the top and bottom plates of each capacitor are equal and opposite. Now if you glance at the region in the gray, dashed box. This box contains essentially a single conductor, separated from the rest of the circuit by insulators. No charge can flow into or out of this region, and the conductor started out electrically neutral. Therefore any negative charge at the top (on the bottom plate of C_1) must equal in magnitude any positive charge at the bottom. Consequently, $q_1 = q_2 = q$, and

$$\frac{q}{C_{\text{eq}}} = \frac{q}{C_1} + \frac{q}{C_2}, \qquad \text{or}$$

$$C_{\text{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}. \tag{2.19}$$

This can be easily generalized to N capacitors in parallel:

$$C_{\text{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)^{-1} = \left(\sum_{n=1}^{N} \frac{1}{C_n}\right). \tag{2.20}$$

An alternative way of arriving at this result, starting with Eq. (2.18) and the definition of current, is to take a derivative:

$$\frac{1}{C_{\text{eq}}}\frac{dq}{dt} = \frac{1}{C_1}\frac{dq_1}{dt} + \frac{1}{C_2}\frac{dq_2}{dt} = \frac{i_1}{C_1} + \frac{i_2}{C_2} = \frac{i}{C_{\text{eq}}}.$$

By KCL, $i = i_1 = i_2$, so the currents all cancel, yielding the same result as before.

Inductors in Series

Inductors in series are shown in Fig. 2.8(c). As should be obvious by now, $v = v_1 + v_2$. By Eq. (1.20), this is

$$L_{\text{eq}}\frac{di}{dt} = L_1\frac{di_1}{dt} + L_2\frac{di_2}{dt}.$$

Since the inductors are in series, the same current flows through both. All currents are equal, so all of the derivatives are equal. They cancel, leaving $L_{\text{eq}} = L_1 + L_2$. The generalization is

$$L_{\text{eq}} = L_1 + L_2 + \ldots + L_N = \sum_{n=1}^{N} L_n. \tag{2.21}$$

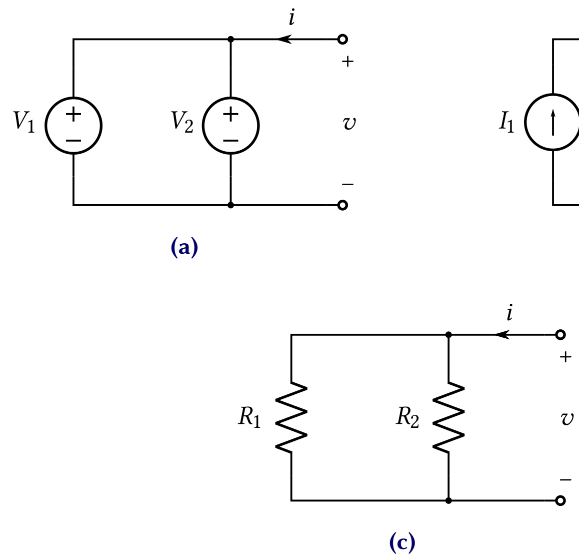


Figure 2.9: Voltage sources, current sources, and resistors in parallel.

2.2.2 Parallel Combinations

Two elements are in **parallel** if their "input" terminals are connected to each other and their "output" terminals are connected to each other. We shall now consider the same elements again.

Voltage Sources in Parallel

Recall that voltage sources maintain a specified voltage between their terminals. In Fig. 2.9(a), we have two voltage sources in parallel, so they are attempting to maintain voltages V_1 and V_2 between their terminals, respectively. But the tops of V_1 and V_2 constitute one node, so they must be at the same potential. At the same time, the bottoms of V_1 and V_2 constitute one node, so they must be at the same potential. As a result, if $V_1 \neq V_2$, one source will push current the "wrong way" through the other.

With the warning given, there are some cases where connecting voltage sources in parallel is sometimes done. For example, if you have a *rechargeable* battery, the fact that the other battery drives current through it the wrong way is actually a good thing—it's what recharges the battery!

Current Sources in Parallel

Fig. 2.9(b) shows two current sources in parallel. Consider the point just above I_2. It has three currents flowing *into* it: i from the outside, I_1 from the leftmost current source, and I_2 from the current source in the center of the figure. By KCL, all currents flowing into or out of a point, junction, element, whatever, must sum (algebraically) to zero. Thus

$$I_1 + I_2 + i = 0.$$

Consequently, the current flowing into the parallel combination of I_1 and I_2 is

$$i = -(I_1 + I_2)\,.$$

Note the negative sign—both current sources point upward, while the input current points i points to the left. Current sources in parallel add algebraically, with minus signs if they point upward and plus signs if they point downward (as drawn).

Resistors in Parallel

Again, resistors in parallel constitute the most common and useful situation in circuit analysis.

Since the tops of the two resistors connect to the same node, they must be at the same potential. The same goes for the bottom of the two resistors. Thus, the voltage drops across the two resistors must be the same, and they must be equal to the voltage v at the output:

$$v_{R_1} = v_{R_2} = v.$$

On the other hand, by KCL, the current i entering the combination must split, so that

$$i_{R_1} + i_{R_2} = i.$$

Rearranging Ohm's law shows that for any given resistor, $i = v/R$, so this becomes

$$\frac{v_{R_1}}{R_1} + \frac{v_{R_2}}{R_2} = i.$$

But, since we've already established that the voltage drops across the resistors are equal, we can write this as

$$\frac{v}{R_1} + \frac{v}{R_2} = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = i, \qquad \text{or}$$

$$v = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} i.$$

Comparing this to Ohm's law reveals that the two parallel resistors can be replaced by a single resistor with resistance

$$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}. \tag{2.22}$$

Like the other relationships we have seen, this can be generalized to N resistors in parallel as

$$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}\right)^{-1} = \left(\sum_{n=1}^{N} \frac{1}{R_n}\right)^{-1}. \tag{2.23}$$

Note that the two-resistor case happens *very* frequently, and it is worth memorizing the simpler result,

$$R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2} \equiv R_1 \parallel R_2, \tag{2.24}$$

which comes simply from taking Eq. (2.22), getting a common denominator, and taking the inverse. The second part defines the notation $R_1 \parallel R_2$, read as "R_1 parallel to R_2." This

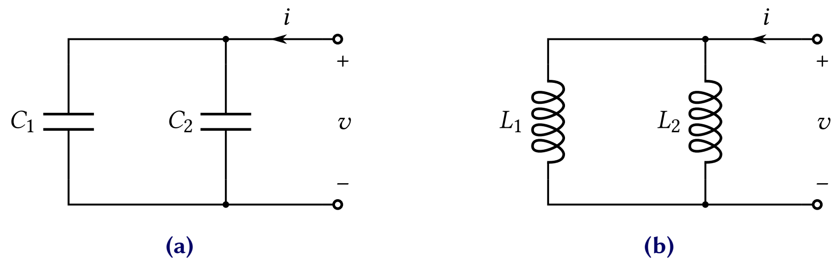


Figure 2.10: Capacitors and inductors in parallel.

formula is easy to remember if you translate it into the words "product over sum[4]." But there's a warning:

> **⚠ Warning**
>
> This simple expression, "product over sum," only works for *two* resistors in parallel. If you have more than two, you need to return to Eq. (2.23). If you are good at spotting cyclic patterns or have studied combinatorics, you may be able to find a simpler approach for more than two resistors, but that is beyond the scope of this text.

Capacitors in Parallel

Two parallel capacitors are shown in Fig. 2.10(a). By KCL $i = i_1 + i_2$. By Eq. (1.17), this becomes

$$C_{\text{eq}} \frac{dv}{dt} = C_1 \frac{dv_1}{dt} + C_2 \frac{dv_2}{dt}.$$

Since the capacitors are in parallel, they must have the same voltage across them, $v = v_1 = v_2$. Thus, the derivatives of the voltages cancel, leaving $C_{\text{eq}} = C_1 + C_2$. For N capacitors all in parallel, this becomes

$$C_{\text{eq}} = C_1 + C_2 + \cdots + C_N = \sum_{n=1}^{N} C_n. \tag{2.25}$$

Inductors in Parallel

Finally, consider the parallel inductors shown in Fig. 2.10(b). By KCL, $i = i_1 + i_2$. Take a time derivative:

$$\frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}.$$

By Eq. (1.20), this can be written

$$\frac{v}{L_{\text{eq}}} = \frac{v_1}{L_1} + \frac{v_2}{L_2}.$$

But by KVL, the voltages are all equal, so they cancel, leaving

$$L_{\text{eq}} = \left(\frac{1}{L_1} + \frac{1}{L_2} \right)^{-1}. \tag{2.26}$$

[4]If you are tempted to confuse "product over sum" with "sum over product," just remember units! The final result must be in ohms, so the thing with Ω^2 must be in the numerator.

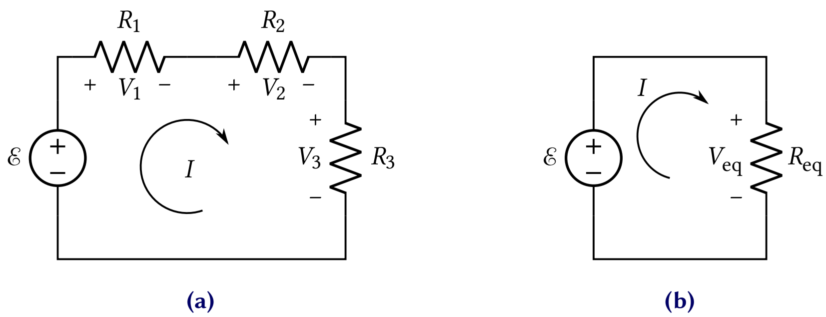


Figure 2.11: Schematic diagram of a simple series circuit containing a voltage source and three resistors.

For N inductors in parallel, we get

$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_2}\right)^{-1}. = \left(\sum_{n=1}^{N} \frac{1}{L_n}\right)^{-1} \tag{2.27}$$

2.2.3 Revisiting Previous Examples

Consider again the circuit in Fig. 2.1, redrawn in Fig. 2.11(a). If our goal is to find the current I supplied by the battery, we can do this problem much more simply than we did it before by noting that all of the resistors are in series. This allows us to replace them all with a single resistor whose resistance is given by Eq. (2.17) as

$$R_{eq} = R_1 + R_2 + R_3.$$

By KVL, the voltage $V_{eq} = \mathcal{E}$, so the current through the equivalent resistor (and, thus, through the battery and the original three resistors) is, by Ohm's law,

$$I = \frac{\mathcal{E}}{R_{eq}} = \frac{\mathcal{E}}{R_1 + R_2 + R_3}. \tag{2.28}$$

2.3 Loop-Current Method

While applying KVL and KLC directly to circuits, perhaps with some simplifications, is always possible in principle, there are a few methods that simplify the process, and usually reduce the number of equations that need to be solved.

The first that we shall discuss is the **loop-current method**, also called the **mesh-current method** since it's usually done with meshes (recall that a mesh is a special kind of loop that does not contain any loops inside it). The result will be a system of equations that can be solved for abstract, linear combinations of currents in the circuit. From these, the actual currents through the branches and, therefore, the voltages at the nodes, can be calculated.

2.3.1 The Loop-Current Procedure

We'll go through the steps of the loop-current method using Fig. 2.12 as an example.

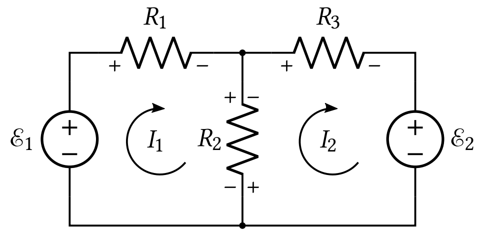


Figure 2.12: A simple example to illustrate the loop current method.

1. Choose your loops. Generally, you should prefer meshes, though, as we'll see, that isn't always possible.
2. Assign each loop a **loop current**, and give it a direction. It does not have to be the actual direction that the current is flowing. A common choice is to make all of them clockwise (as we've done here). Another common choice is to alternate directions, which results in fewer minus signs in the final system of equations.
3. Go around each loop *in the direction of the loop current*, and mark the polarities of your elements.
   - Ideal voltage sources do not need to be marked, since their circuit symbol already indicates polarity.
   - Ideal current sources do not need to be marked; we'll have more to say about current sources shortly.
   - Resistors get a + at the terminal where the loop current enters and a − where it exits. Since you're traversing the loop in the direction of the loop current, resistors get a + at the first terminal you reach.
   - Batteries get a + on the long line and a − on the short line.
4. For each loop, write down a loop-current equation using the element relations for each element. Pick some arbitrary starting point, go around the loop in the direction of the loop current, and sum either all of the voltage rises or all of the voltage drops until you get back to your starting point. If multiple loop currents pass through a single device, the loop current you write will be the *algebraic sum* of the loop currents. Set the result to zero.

   For our example circuit, if we start in the bottom-left current in each, we get, summing the rises,

$$\mathcal{E}_1 - R_1 I_1 - R_2(I_1 - I_2) = 0 \qquad \text{and}$$

$$-R_2(I_2 - I_1) - R_3 I_2 - \mathcal{E}_2 = 0.$$

   Note the signs of the currents through R_1. When we write the loop current equation for the first loop, the current goes down through the resistor, so we are traversing a drop. But the *total* current through that resistor is the *algebraic sum* of the loop currents through it, and we've also got I_2 going up. So the total current *down* is $I_1 - I_2$.
5. Rearrange the equations to find

$$(R_1 + R_2)I_1 - R_2 I_2 = \mathcal{E}_1, \qquad \text{and}$$

$$-R_2 I_1 + (R_2 + R_3)I_2 = -\mathcal{E}_2.$$

6. This is a system of two equations in two unknowns, and it can be solved straightforwardly using your favorite method. A review of Cramer's method is given in the Appendix. The result is

$$I_1 = \frac{\mathcal{E}_1(R_2 + R_3) - \mathcal{E}_2 R_2}{R_1R_2 + R_1R_3 + R_2R_3} \tag{2.29a}$$

$$I_2 = \frac{\mathcal{E}_1 R_2 - \mathcal{E}_2(R_1 + R_2)}{R_1R_2 + R_1R_3 + R_2R_3}. \tag{2.29b}$$

Note that this finding is general. If you have N independent loops, you will find N independent equations in N unknowns. Eq. (2.29)

Current Sources in the Loop-Current Method If your circuit contains current sources, step 4 needs to be modified depending on where the current source is.

1. If the current source is placed so that it is in *only one* loop, great! You already know one of the loop currents.
2. If the current source is shared between two loops, you cannot use only meshes. You must write down loop equations that *go around* the current source, and then equate the rated source current to the algebraic sum of the loop currents passing through it.

Example 2.2

Consider the circuit of Fig. 2.13. Write down the loop current equations that could be solved for the currents.

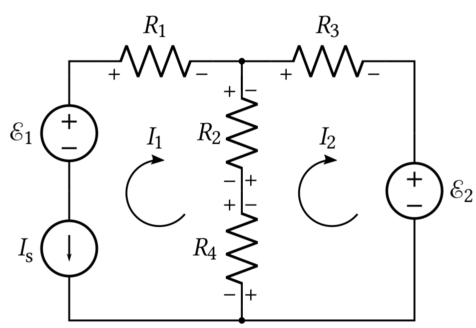


Figure 2.13: A circuit with a current source on the edge of the loop.

Solution: The loop equation on the right is found by the same procedure as before. Start at the bottom left corner, and this time sum the *drops*:

$$R_4(I_2 - I_1) + R_2(I_2 - I_1) + R_3I_3 + \mathcal{E}_2 = 0.$$

On the left, however, we have a current source. Fortunately, it falls under case 1, so we immediately know the loop current: $I_1 = -I_s$. The negative comes from the fact that the source pushes current down, while our arbitrary loop current goes up through it. In this case, we only need *one* loop equation!

Example 2.3

Now put the current source in the middle, as in Fig. 2.14. Write down a system of equations that could be used to find the loop currents.

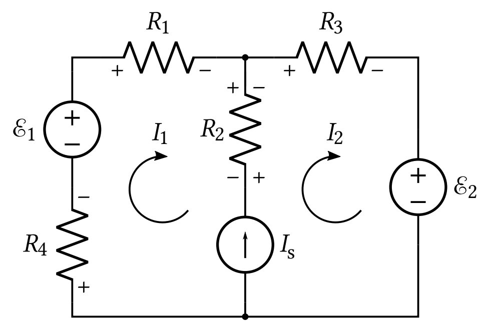


Figure 2.14: A circuit with a current source shared between two meshes.

Solution: We cannot write down a loop equation that includes the current source, so we have to go around it. Going around the outer loop, summing the drops, we get

$$R_4 I_1 - \mathcal{E}_1 + R_1 I_1 + R_3 I_2 + \mathcal{E}_2 = 0.$$

Notice that, as we go around the outer loop, we *stick with the currents and polarities defined in steps 2 and 3* of the general procedure. Now we have one equation in two unknowns. We need another, and it comes from the current source. I_1 flows down the center path, I_2 flows up it, and, by the nature of a current source, a total current of I_s flows up. Therefore,

$$I_s = I_2 - I_1.$$

Now we have enough to determine both of the loop currents.

2.3.2 Using the Loop Currents

Now that we've seen how to determine the loop currents, let's examine what they actually mean. The best way to do that is by example, so we'll return to the circuit from Fig. 2.12, the loop currents of which were expressed in Eq. (2.29), repeated here:

$$I_1 = \frac{\mathcal{E}_1(R_2 + R_3) - \mathcal{E}_2 R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$
$$I_2 = \frac{\mathcal{E}_1 R_2 - \mathcal{E}_2(R_1 + R_2)}{R_1 R_2 + R_1 R_3 + R_2 R_3}.$$

Let's examine the results by choosing some values. First, we'll take $R_1 = R_2 = R_3 = 100\ \Omega$, $\mathcal{E}_1 = 5$ V, and $\mathcal{E}_2 = 1$ V. In that case, Eq. (2.29) yield $I_1 = 30$ mA and $I_2 = 10$ mA. The first point is that both of these numbers are positive, which means that our guesses for the loop-current directions were correct. Since only one loop current passes through R_1, the total current through R_1 is just the loop current: $I_{R_1} = 30$ mA. Likewise, $I_{R_3} = I_2 = 10$ mA. The current through R_2 is a linear combination of the two loop currents; the current downward is $I_{R_2} = I_1 - I_2 = 30\text{ mA} - 10\text{ mA} = 20$ mA. Once we have the current through the three resistors, we can find the voltages across them using Ohm's law.

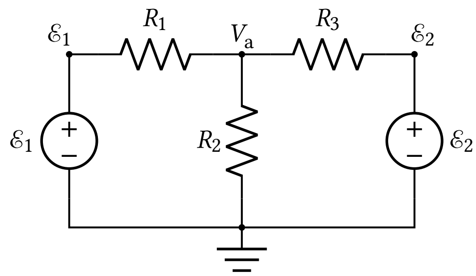


Figure 2.15: A simple circuit to study the node-voltage method.

On the other hand, we can let $\mathcal{E}_1 = \mathcal{E}_2 = 3$ V, holding the resistor values all at 100 Ω. Then Eq. (2.29) yield $I_1 = 10$ mA and $I_2 = -10$ mA. I_1 is positive, so our guess was correct for the left loop current. The current through R_1 is 10 mA to the right. But I_2 is negative, so our guess for the right-hand loop current was incorrect. Current is traveling *upward* through $\mathcal{E}_2$. The current through R_2 is still a linear combination; its new value is $I_{R_2} = I_1 - I_2 = 10\text{ mA} - (-10\text{ mA}) = 20$ mA. The voltage across R_2 is given by Ohm's law as $V_2 = I_{R_2} R_2 = (20\text{ mA})(100\ \Omega) = 2$ V.

2.4 Node-Voltage Method

Whereas the loop-current method focused on using KVL to find the currents through the loops in a circuit, the **node-voltage method** uses Kirchhoff's current law to determine the voltages at the nodes in the circuit. Similar to the loop-current method, it will result in a system of equations. However, these equations can then be solved for the voltages at various points in the circuit, rather than the currents through branches.

2.4.1 The Node-Voltage Procedure

Again, we will explain the procedure using an example, shown in Fig. 2.15.

1. Choose one point to be the zero-point—ground—of electric potential (voltage), and mark it. The particular point you choose is not important, though there are some points that make the analysis easier. Typically, if you can choose a point which is at the negative terminal of all or most power (current and/or voltage) sources, that's a good first guess.

   In our example, the bottom of the circuit is a good choice.
2. Give labels to the voltages at all other nodes. If you have nothing but a voltage source in a branch, with ground at the other end, immediately label the node with the voltage of the source. Otherwise, assign a variable name.

   In our example, the left-most and right-most nodes are determined simply by voltage sources, leaving only the upper-middle node needing a name. In this case, we call is v_a.
3. For the unknown node voltages, write a KCL equation, summing the currents flowing out or the currents flowing in (using Ohm's law or whatever element relationships are appropriate), and set the algebraic sum to zero. If we sum the currents out

in our example we have

$$\frac{V_a - \mathcal{E}_1}{R_1} + \frac{V_a - 0\text{ V}}{R_2} + \frac{V_a - \mathcal{E}_2}{R_3} = 0.$$

We have used Ohm's law, which states that the current through any resistor is $\Delta V/R$, where $\Delta V = V_{high} - V_{low}$. Since we're summing currents flowing *out*, we assume that V_a is the "high" voltage. If this assumption is wrong, we'll get a negative in the result, and all will be well. Do the same at every node—if you are summing the current flowing out, take the voltage at the node you are considering as "high." Note that in the middle term, we have $V_a - 0$ V, indicating that the current through R_2 flows out to ground. In the future, we will not explicitly write the −0 V.

4. Combine coefficients of variables:

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) V_a = \frac{\mathcal{E}_1}{R_1} + \frac{\mathcal{E}_2}{R_3}.$$

5. Now, solve for V_a:

$$V_a = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} \left(\frac{\mathcal{E}_1}{R_1} + \frac{\mathcal{E}_2}{R_3}\right) = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \frac{R_3 \mathcal{E}_1 + R_1 \mathcal{E}_2}{R_1 R_3}$$

$$V_a = \frac{R_2(R_3 \mathcal{E}_1 + R_1 \mathcal{E}_2)}{R_1 R_2 + R_1 R_3 + R_2 R_3}. \tag{2.30}$$

In general, for a circuit with N nodes, we will get $N - 1$ independent equations in $N - 1$ unknowns. The −1 comes from choosing one of the nodes to be ground. In the circuit we just analyzed, there were 4 nodes, so we would expect 3 equations. However, since two nodes were separated from ground by a voltage source, the equations for those nodes were trivial; they were substituted implicitly in step 2.

Supernodes If the circuit contains a voltage source between two non-ground nodes, you have to analyze a **supernode**, where you sum the current entering or leaving the *pair* of nodes separated by the voltage source. This reduces the number of equations we get using the standard procedure, but we can get an extra one by noting that the rated voltage of the source gives us an additional equation relating the two neighboring voltages.

Example 2.4

Consider the circuit of Fig. 2.16. Write down the a system of equations which could be solved for all of the node voltages.

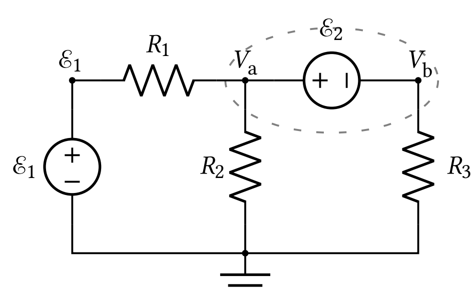


Figure 2.16: A circuit which requires a supernode.

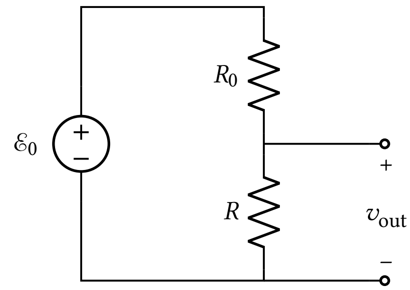


Figure 2.17: A voltage divider.

Solution: We have already followed steps 1 and 2 of the general procedure, but there is some difficulty with nodes a and b. We have to write one equation for the currents leaving the *combination* of the two nodes:

$$\frac{V_a - \mathcal{E}_1}{R_1} + \frac{V_a}{R_2} + \frac{V_b}{R_3} = 0.$$

Now we have one equation with two unknowns. We can get a third equation using the element relation for the voltage source between the two nodes: $V_a = V_b + \mathcal{E}_2$. We can now easily solve for all unknown quantities.

2.4.2 Using the Node Voltages

As in the loop-current case, we will consider this in terms of an example, considering the circuit shown in Fig. 2.15, the analysis of which is summarized by Eq. (2.30), repeated here:

$$V_a = \frac{R_2(R_3\mathcal{E}_1 + R_1\mathcal{E}_2)}{R_1R_2 + R_1R_3 + R_2R_3}.$$

If we use the same values as in the loop-current case ($R_1 = R_2 = R_2 = 100\ \Omega$, along with $\mathcal{E}_1 = 5$ V and $\mathcal{E}_2 = 1$ V), we find $V_a = 2$ V. The current to the left through R_1 is then $(V_a - \mathcal{E}_1)/R_1 = (2\text{ V} - 5\text{ V})/(100\ \Omega) = -30$ mA, meaning that $I_{R_1} = 30$ mA to the right.

2.5 A Pattern: The Voltage Divider

The **voltage divider**, as shown in Fig. 2.17, is one of the most common series-circuit patterns. It happens all the time. We will analyze it assuming that the load is insignificant, so no current goes through the v_{out} terminals. Then the whole current through the circuit is

$$I = \frac{\mathcal{E}_0}{R_0 + R},$$

and the voltage across the “output resistor” is $v_{out} = IR$, so

$$v_{out} = \mathcal{E}_0 \frac{R}{R_0 + R}. \tag{2.31}$$

This is one of the most often used equations in electronics. Although memorization is usually not recommended, make an exception for this one. Remember not only the equation but the procedure for obtaining it.

If the voltage divider has more than two resistors, the equation is still straightforward to construct. The numerator is the sum of all the resistors across which the output is taken, and the denominator is the sum of all resistors in the divider chain. The resistor ratio is multiplied by the voltage across the whole chain.

Example 2.5

Consider the voltage divider in Fig. 2.18. Assume $V_{\text{in}} = 100$ V. Determine the output voltages across terminals (a) 1–0, (b) 3–0, and (c) 5–0.

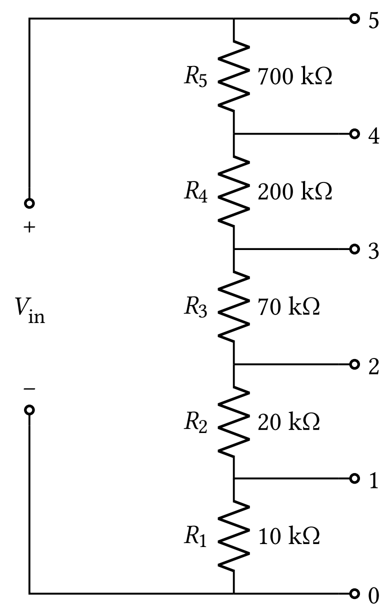


Figure 2.18: A voltage divider chain.

Solution: The denominator is the sum of all of the resistors, which will be the same for all three parts of the problem:

$$R_{\text{tot}} = 10\text{ k}\Omega + 20\text{ k}\Omega + 70\text{ k}\Omega + 200\text{ k}\Omega + 700\text{ k}\Omega = 1000\text{ k}\Omega.$$

(a) $R_{1-0} = R_1 = 10$ kΩ, so

$$V_{1-0} = V_{\text{in}} \frac{R_1}{R_{\text{tot}}} = (100\text{ V}) \left(\frac{10\text{ k}\Omega}{1000\text{ k}\Omega} \right) = 1\text{ V}.$$

(b) If we are instead measuring positive voltage at terminal 3, we have $R_{3-0} = R_1 + R_2 + R_3 = 10\text{ k}\Omega + 20\text{ k}\Omega + 70\text{ k}\Omega = 100\text{ k}\Omega$. Thus

$$V_{3-0} = V_{\text{in}} \frac{R_{3-0}}{R_{\text{tot}}} = (100\text{ V}) \left(\frac{100\text{ k}\Omega}{1000\text{ k}\Omega} \right) = 10\text{ V}.$$

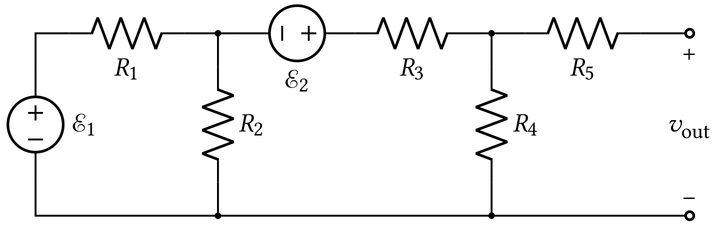


Figure 2.19: A complicated network with output terminals.

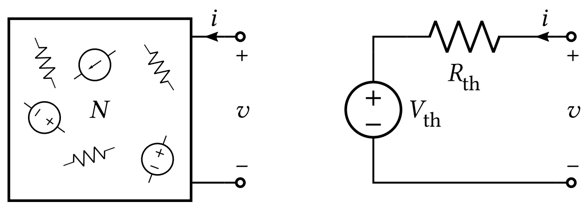


Figure 2.20: A complex (unspecified) network of linear elements along with its Thevenin equivalent.

(c) Now we measure the voltage across terminals 5 and 0, so $R_{5-0} = R_{tot} = 1000\ \text{k}\Omega$, and

$$V_{5-0} = V_{in}\frac{R_{tot}}{R_{tot}} = 100\ \text{V}$$

2.6 Thevenin and Norton Equivalents

Most of the circuits we have seen so far have been self-contained. It might seem to you as if they are of limited usefulness; you're right. More useful are circuits which, like the voltage divider, have *output terminals*; such circuits can be used to deliver electrical energy to *other* circuits. Such another circuit which is being driven by the circuit at hand is called the **load**.

Consider the rather complicated circuit shown in Fig. 2.19. Assuming we want to know the power that such a circuit would deliver to some load resistor, the procedure would be straightforward. We would draw the load resistor across the terminals and use either the loop-current or node-voltage method to develop and then solve a system of three equations for the current through and voltage across the load.

But what if the load can be swapped out for a different resistor? Or a more complicated circuit? Any change you make will require you to solve your node-voltage or loop-current equations again. That can get very tedious very quickly. But there's a solution!

2.6.1 Thevenin's Theorem

Thevenin's theorem states that any network of voltage sources, current sources, and resistors—no matter how complicated—can be replaced by a single voltage source in series with a single resistor, as schematically illustrated in Fig. 2.20.

By KVL, we can see that

$$v = V_{th} + iR_{th}.$$

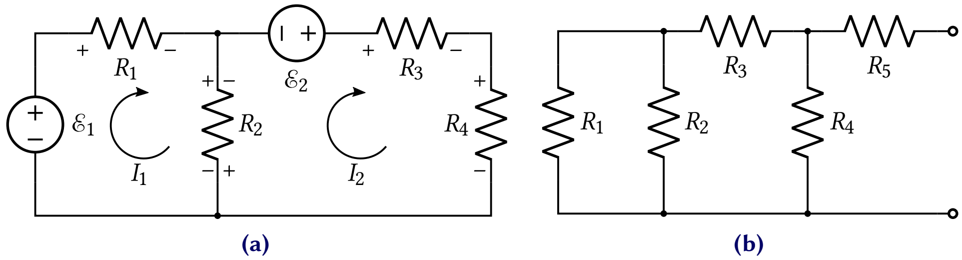


Figure 2.21: Circuits for determining the Thevenin equivalent voltage and resistance of the circuit in Fig. 2.19.

We can see, then, that the Thevenin voltage

$$V_{\text{th}} = V_{\text{open}} \tag{2.32}$$

is the open-circuit voltage of the network (i.e. the voltage across the terminals if there is no current path between them, so that $i = 0$). The Thevenin resistance R_{th} is found by setting $v = 0$ (i.e. shorting the terminals so that $i = I_{\text{short}}$) and rearranging to find

$$R_{\text{th}} = -\frac{V_{\text{open}}}{I_{\text{short}}} = -\frac{V_{\text{th}}}{I_{\text{short}}}. \tag{2.33}$$

The minus sign comes from the fact that, as the circuit is drawn, the short-circuit current would flow clockwise—opposite the marked direction of i—and, therefore, be negative.

An equivalent (and often easier) way to determine R_{th} is to calculate equivalent resistance of N with all sources turned off. Turning a supply "off" means setting it's rated value to zero: for a voltage source, $V = 0$ means a short circuit; for a current source, $I = 0$ means an open circuit.

Let's find expressions for these quantities for the circuit in Fig. 2.19. First, we'll find the Thevenin, or open-circuit, voltage. The open-circuit condition means no current can pass through R_5, so we can remove it from the drawing. The circuit, prepared for the loop current method, is redrawn in Fig. 2.21(a). Our loop equations are

$$-\mathcal{E}_1 + R_1 I_1 + R_2(I_1 - I_2) = 0 \qquad \Longrightarrow \qquad (R_1 + R_2)I_1 - R_2 I_2 = \mathcal{E}_1 \qquad \text{and}$$

$$R_2(I_2 - I_1) - \mathcal{E}_2 + R_3 I_2 + R_4 I_2 = 0 \qquad \Longrightarrow \qquad -R_2 I_1 + (R_2 + R_3 + R_4)I_2 = \mathcal{E}_2.$$

These can be solved to find I_2, and the Thevenin voltage is then simply the voltage across R_4:

$$V_{\text{th}} = R_4 I_2 = \frac{R_4\left[R_2\mathcal{E}_1 + (R_1 + R_2)\mathcal{E}_2\right]}{(R_1 + R_2)(R_2 + R_3 + R_4) - R_2^2}. \tag{2.34}$$

To find the Thevenin resistance, we "turn off" all power supplies, replacing voltage sources with wire, as shown in Fig. 2.21(b). We can work from left to right, using Eqs. (2.17) and (2.24) to reduce series and parallel combinations. R_1 and R_2 are in parallel with each other. That combination is in series with R_3. That combination is in parallel with R_4. That combination is in series with R_5. Thus:

$$R_{\text{th}} = R_5 + R_4 \parallel \left[R_3 + (R_1 \parallel R_2)\right],$$

which half a page of algebra turns into this pleasantly simple expression:

$$R_{\text{th}} = \frac{R_5\left[(R_1+R_2)(R_3+R_4)+R_1R_2\right]+R_4\left[(R_1+R_2)R_3+R_1R_2\right]}{(R_1+R_2)(R_3+R_4)+R_1R_2}. \tag{2.35}$$

Well, that was quite a bit of work (normally it will be a little less, since you will have the values of the resistors and the voltage sources, so you can combine values as you go along, working only with simple equations). You might ask, "What's the use?" The answer is that—for this circuit—you never have to do it again. If you put a load resistor across the output terminals, the circuit you have to analyze is a single, simple loop consisting of a voltage source and two resistors in series.

Example 2.6

Find Thevenin's equivalent circuit for the circuit of Fig. 2.19 if $\mathcal{E}_1 = 30$ V, $\mathcal{E}_2 = 3.0$ V, $R_1 = 240\ \Omega$, $R_2 = 400\ \Omega$, $R_3 = 350\ \Omega$, $R_4 = 750\ \Omega$, and $R_5 = 100\ \Omega$. Then attach a load resistor R_L across the output. Determine (a) the power dissipated by the load if $R_L = 1$ kΩ, (b) the power dissipated by the load if $R_L = 400\ \Omega$, (c) the value of R_L that will dissipate the maximum power.

Solution: We've already worked out expressions for the Thevenin voltage and resistance. Simply plug the values into Eqs. (2.34) and (2.35) to find $V_{\text{th}} = 13.05$ V and $R_{\text{th}} = 400\ \Omega$. Our new circuit, with the load resistor in place, is

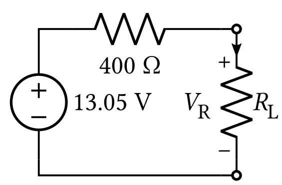


Figure 2.22: The resulting Thevenin equivalent circuit.

(a) Our new circuit is simply a voltage divider. The voltage across our load is given by Eq. (2.31) as

$$V_R = V_{\text{th}}\frac{R_L}{R_L + R_{\text{th}}} = (13.05\text{ V})\left(\frac{1\text{ k}\Omega}{1\text{ k}\Omega + 400\ \Omega}\right) = 9.32\text{ V}.$$

The power dissipated is given by Eq. (1.13) as

$$P = \frac{V_R^2}{R_L} = \frac{(9.32\text{ V})^2}{1\text{ k}\Omega} = 86.9\text{ mW}.$$

(b) Using the same procedure as in (a), but with 400 Ω instead of 1 kΩ, we find $P = 106$ mW. Thanks to Thevenin, this required practically no extra work!

(c) We can combine the voltage-divider and power equations to write power as

$$P = \frac{V_{\text{th}}^2 R_L}{(R_{\text{th}} + R_L)^2}.$$

Differentiate, and set the result equal to zero:

$$\frac{dP}{dR_L} = \frac{V_{th}^2}{(R_L + R_{th})^2} - \frac{2R_L V_{th}^2}{(R_L + R_{th})^3} = 0 \qquad \Longrightarrow \qquad R_L = R_{th}.$$

Thus, to dissipate maximum power, we need $R_L = 400\ \Omega$.

Example 2.7

When a function generator has no load connected, its output voltage is 7.87 V. When a 100 Ω resistor is connected as a load, the voltage across it is 5.20 V. What is the output resistance of the function generator?

Solution: We treat the function generator as its Thevenin equivalent. The equivalent voltage source is always equal to the open-circuit output voltage, so $V_{th} = 7.87$ V. The voltage across the load (which we are told is 5.20 V) is given by Eq. (2.31):

$$V_R = V_{th} \frac{R_L}{R_L + R_{th}}.$$

Solve for the source's output resistance, which is its Thevenin equivalent resistance:

$$R_{th} = R_L \left(\frac{V_{th}}{V_R} - 1 \right) = (100\ \Omega) \left(\frac{7.87\ \text{V}}{5.20\ \text{V}} - 1 \right) = 51.3\ \Omega.$$

Thevenin's theorem can be used to

1. solve for the load conditions in circuits which have output terminals;
2. find the output resistance of devices, such as signal generators, even when we do not know about their internal circuitry; and
3. simplify circuits by allowing a key resistor to be removed (we have not done an example of this, but we will see how it works with Norton equivalent circuits shortly).

This is why it can be said that Thevenin's theorem is one of the most useful theorems in circuit analysis.

2.6.2 Norton's Theorem

Norton's theorem states that any network voltage sources, current sources, and resistors—no matter how complicated—can be replaced by a single current source in parallel with a single resistor, as shown in Fig. 2.23. From KCL, we can see that

$$i = \frac{v}{R_n} - I_n$$

If we set $v = 0$, we can see that the Norton equivalent current is the negative of the short-circuit current:

$$I_n = -I_{short}. \tag{2.36}$$

On the other hand, if we set $i = 0$ and determine the open-circuit voltage, we find that

$$R_n = \frac{V_{open}}{I_n} = -\frac{V_{open}}{I_{short}}. \tag{2.37}$$

Compare Eqs. (2.33) and (2.37) to see that $R_{th} = R_n$, so the alternate method we used to calculate the Thevenin resistance also works to calculate the Norton resistance.

Example 2.8

Take $\mathcal{E}_1 = 12$ V, $\mathcal{E}_2 = 6$ V, $R_1 = 330\ \Omega$, $R_2 = 75\ \Omega$, and $R_3 = 100\ \Omega$. Use Norton's theorem to determine the voltage drop across R_2 in the circuit.

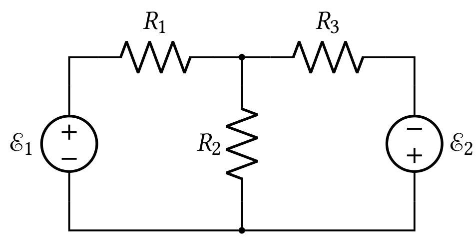


Figure 2.24: A circuit to solve using Norton's theorem.

Solution: We want to find the Norton equivalent of the circuit *surrounding* R_2.

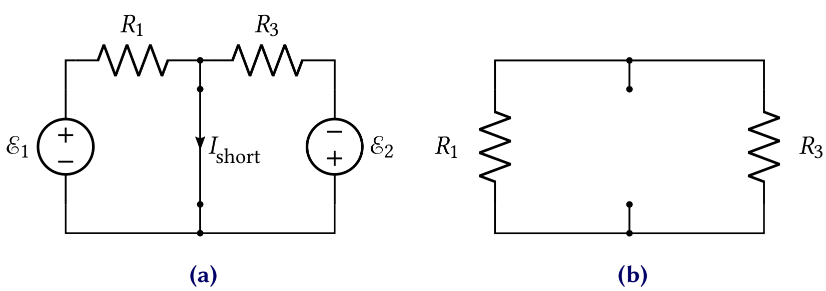


Figure 2.25: Circuits for finding the Norton equivalent current and resistance.

To do that, we first need to find the short-circuit current. In Fig. 2.25(a), we have two loops. The left causes a current downward through the center of $I_1 = \mathcal{E}_1/R_1 = (12\text{ V})/(330\ \Omega) = 36.36$ mA. The right loop causes a currend upward through the center of $I_2 = \mathcal{E}_2/R_3 = (6\text{ V})/(100\ \Omega) = 60.00$ mA. Thus, the total short-circuit current (given the direction labeled) is $I_{short} = (36.36\text{ mA}) - (60.00\text{ mA}) = -23.64$ mA. Because it is negative, we conclude that the current flows upward.

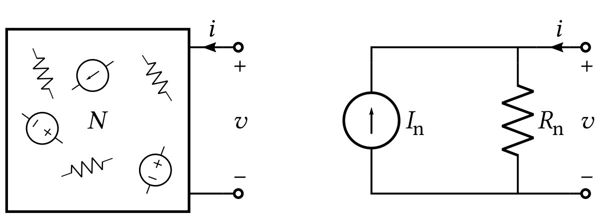


Figure 2.23: A complex (unspecified) network of linear elements along with its Norton equivalent.

To find the Norton equivalent resistance, we consult Fig. 2.25(b), in which R_2 has been replaced with an open circuit. The equivalent resistance of what remains is

$$R_\text{n} = R_1 \parallel R_3 = \frac{R_1 R_3}{R_1 + R_3} = \frac{(330\ \Omega)(100\ \Omega)}{(330\ \Omega) + (100\ \Omega)} = 76.74\ \Omega.$$

Rather than trying to think rigorously about the signs in Eq. (2.36), we can instead just do what makes sense. To make the current flow upward through R_2, we need the current to flow out of the bottom terminal of the current source. Therefore, we have

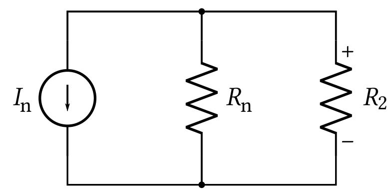


Figure 2.26: The Norton equivalent circuit.

We have a total current I_n flowing up through the parallel combination of R_n and R_2. Both resistors must drop the same voltage, since they are in parallel. Therefore, the absolute value of the voltage across R_2 is

$$|V_2| = I_\text{n}(R_\text{n} \parallel R_2) = 23.64\ \text{mA}\frac{(75\ \Omega)(76.74\ \Omega)}{(75\ \Omega) + (76.74\ \Omega)} = 0.897\ \text{V}.$$

Since we know that the current flows up, we conclude that the bottom terminal of R_2 is at higher voltage. With the polarities as marked, then, $V_2 = -0.897$ V.

Exercises and Problems

1. In Fig. 2.1, $\mathcal{E} = 6.60$ V, $R_1 = 33\ \Omega$, $R_2 = 22\ \Omega$, and $R_3 = 11\ \Omega$. How much current is flowing in the circuit?
2. In Fig. 2.1, $\mathcal{E} = 7.20$ V, $R_1 = 470\ \Omega$, $R_2 = 220\ \Omega$ and $R_3 = 100\ \Omega$. What are
   (a) the voltage across R_1,
   (b) the voltage across R_2, and
   (c) the voltage across R_3?
3. In Fig. 2.1, $\mathcal{E} = 22.5$ V, $R_1 = 3.9\ \text{k}\Omega$, $R_3 = 4.7\ \text{k}\Omega$, $V_2 = 7.500$ V, and $V_3 = 8.198$ V. What is the resistance R_2?
4. In Fig. 2.3, $\mathcal{E} = 6.60$ V, $R_1 = 33\ \Omega$, $R_2 = 22\ \Omega$, and $R_3 = 11\ \Omega$. How much current is flowing in the circuit?
5. In Fig. 2.3, the battery current $I = 200$ mA, $R_1 = 470\ \Omega$, $R_2 = 220\ \Omega$ and $R_3 = 100\ \Omega$. What are
   (a) the current through R_1,
   (b) the current through R_2, and
   (c) the current through R_3?
6. In Fig. 2.3, $R_1 = 2.2\ \text{k}\Omega$, $R_2 = 1.5\ \text{k}\Omega$, and $R_3 = 3.6\ \text{k}\Omega$. If $I_1 = 8.182$ mA, what is I_3?
7. In Fig. 2.3, $I = 5.987$ mA, $R_1 = 3.9\ \text{k}\Omega$, $R_3 = 4.7\ \text{k}\Omega$, $I_2 = 1.765$ mA, and $I_3 = 1.915$ mA. What is the resistance R_2?

8. In the circuit of Figure 0.10 if $R = 1\ \text{M}\Omega$, $C = 1\ \mu\text{F}$ and $\mathcal{E} = 10.0$ V. The capacitor starts out fully discharged. How long after the switch is thrown to position 2 does the voltage across the capacitor reach 9.0 V?
9. A time delay circuit triggers when the voltage reaches 2/3 of the applied voltage. The capacitor always starts charging from 0 V. It is desired that the circuit will trigger 1.5 s after the timing cycle begins. If the capacitor's capacitance is 0.1 μF, what value resistor is required in the circuit?
10. Starting with equation 0.57 and using only Ohm's and Kirchhoff's laws, show that the voltage across an inductor is given by

$$v_{\text{L}} = \mathcal{E}\, e^{-tR/L.}$$

11. The current through a 50 mH inductor is to be changed from some value I_{L} to zero in a time of 5 μs. What must the value of I_{L} be to give a 20 kV voltage spike?
12. What is the time constant of a coil which has 8 H of inductance and 40 Ω of resistance?
13. A relay is a device which employs an electromagnet to close one or more sets of switch contacts. The switch contacts can carry a much larger current than is required to energize the electromagnet. The electromagnet is a coil of wire which has inductance and resistance. A given relay is taking too long to close its contacts in a particular application. It has been determined that the delay is not due to mechanical inertia of the moving parts but is due to the time constant of the *RL* circuit of the coil. Discuss what may be done to shorten the time required for the relay to close. The relay itself may not be replaced or altered.

Chapter 3

Alternating Current Circuits

3.1 Alternating Current

To date, we have dealt with sources of electrical energy which deliver direct current, or **dc current**, which flows continuously in one direction. On the other hand, alternating current, or **ac current**, is ubiquitous in our world. The energy provided by the electric company and audio, radio, and television signals are all ac.

Alternating current reverses its direction at regular time intervals. The current flows for a certain period of time in one direction, then reverses and flows for some time, and then reverses again. There is no net movement of electrons in a wire carrying ac; the electrons simply vibrate in place.

A Note on Terminology The term "ac" is often used in ways which seem to be redundant or even contradictory. For example you might read "ac current…" or "ac voltage" or even "ac power." But you might ask, "Doesn't 'ac' stand for 'alternating current'?" If so, the first example is redundant, "alternating current current," while the next two are contradictory. In spite of the redundant or contradictory nature of these terms, their usage is too pervasive to be avoided even in formal writing. The same goes for "dc."

Also note that, throughout this text, "ac" and "dc" are always lowercase, even in titles, while in many other books they are capitalized. This is the convention of both the American Institute of Physics and the Institute of Electrical and Electronics Engineers.

The Cycle A cycle is defined as follows. The current (and/or voltage) rises from zero to a positive maximum, falls back to zero and "rises" to a negative "maximum" and then returns to zero. This cycle then repeats, over and over.

The actual *shape* of the rises and falls—the functional form of the periodic pattern of repeating cycles—is called the **waveform**. Four common waveforms are shown in Fig. 3.1; the most common is the sine wave of Fig. 3.1(a). There are infinitely many possible waveforms, which is best depends on the application.

Period and Frequency The **period** T of an ac current is the time required to complete one cycle, and the frequency f is the number of cycles which occur each second. The period is measured in units of time such as seconds, milliseconds, or microseconds. The frequency has units of cycles/second or 1/s ("cycles" is not really a unit—the word stands in for the number of repetitions of the pattern and is therefore dimensionless). The unit of frequency is called the hertz (1 Hz = 1/s).

The relationship between period and frequency is

$$T = \frac{1}{f}$$

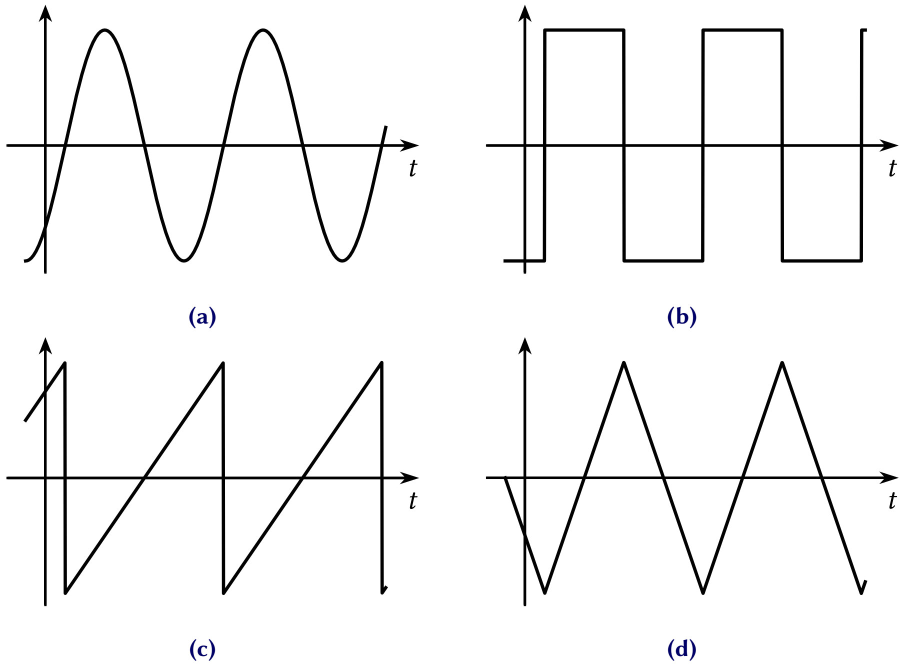


Figure 3.1: Four different, common waveforms. The vertical axis can represent any time-varying quantity, though for our purposes it will nearly always be either current or voltage.

Example 3.1

The frequency of the ac power that comes from wall outlets is 60 Hz. What is the period?

Solution:

$$T = \frac{1}{f} = \frac{1}{60\text{ Hz}} = 1.67 \times 10^{-2}\text{ s} = 16.7\text{ ms}$$

Wavelength and Frequency The only time that the term "wave" may strictly be applied to an alternating current is when the current has been converted into a periodic disturbance which propagates through some medium. For example, a loudspeaker converts ac energy of appropriate frequency into sound waves which propagate through the air; if ac of an appropriate frequency flows in an antenna, electromagnetic waves will be radiated which will propagate through air or the vacuum of space.

When waves propagate through a physical medium with speed v, each cycle of the wave has a physical length, called the **wavelength**, given by

$$\lambda = \frac{v}{f}.$$

The speed of sound in air is approximately 335 m/s, and the speed of electromagnetic waves in vacuum is 3.00×10^8 m/s.

Example 3.2

A sound wave in air has a wavelength of 12 cm. What is its frequency?

Solution: Using the speed of sound, we have

$$f = \frac{v}{\lambda} = \frac{335\ \text{m/s}}{0.12\ \text{m}} = 2790\ \text{Hz}.$$

Example 3.3

A radio station is transmitting on a frequency of 88.9 MHz. What is the wavelength of the electromagnetic wave being transmitted?

Solution: Using the speed of light, we have

$$\lambda = \frac{v}{f} = \frac{3.00 \times 10^8\ \text{m/s}}{88.9 \times 10^6\ \text{Hz}} = 3.37\ \text{m}.$$

Angular Frequency When working through calculations involving ac, it is often necessary for the frequency to appear in the argument of a sine or cosine function. A particular point on a waveform can be found by multiplying the frequency by the amount of time which has passed since the waveform has started. This would be all right if cycles were the only thing we had to deal with. The arguments of sines and cosines must be angles, not cycles.

Since each cycle ends and begins at the same point, we can think of them like complete revolutions, so one cycle consists of 360° or 2π rad. So frequency expressed in angular units—specifically rad/s—called the **angular frequency** ω (lowercase Greek letter "omega") is given by

$$\omega = 2\pi f. \tag{3.1}$$

To illustrate, we can express a sinusoidal current as a function of time t as $i = I \sin(2\pi f t)$. The 2π ensures that when $t = T = 1/f$, the waveform has returned to its starting ($t = 0$) point, so one period does, indeed, represent the time for one cycle to complete. This can be written more compactly using the angular frequency:

$$i(t) = I \sin(\omega t). \tag{3.2}$$

Specifying the Magnitude of ac Signals Consider the alternating current represented by Eq. (3.2). The peak value, or **amplitude**, of the current is I. The **peak-to-peak** current is $2I$. These—peak current or current amplitude and peak-to-peak current—are the most obvious ways of specifying how much current there is, but there is another common and important way.

Suppose you wanted to know the power dissipated by a resistor passing an ac current. We could use the familiar $p = i^2/R$, but this is complicated, because i is changing! What value of current should we use? What we really care about is the *average* power P absorbed by the resistor over each cycle, which we can get by integrating the instantaneous power and dividing by the period:

$$P = \frac{1}{T}\int_0^T p_{\mathrm{a}}(t)\,dt = \frac{R}{T}\int_0^T i^2(t)\,dt = \frac{I^2R}{T}\int_0^T \sin^2\omega t\,dt.$$

Using the trigonometric identity Eq. (A.10), we can rewrite this as

$$P = \frac{I^2R}{T}\int_0^T \frac{1}{2}(1-\cos 2\omega t)\,dt = \frac{I^2R}{2T}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T = \frac{I^2R}{2T}(2T) = \frac{1}{2}I^2R.$$

Thus, we can see that ac current of amplitude I dissipates half the power that a steady dc current of the same amplitude would. If we define the **root-mean-square** (rms) current

$$I_{\mathrm{rms}} = \frac{I}{\sqrt{2}}, \tag{3.3}$$

we can then write the average power as

$$P = I_{\mathrm{rms}}^2 R.$$

In summary, the power dissipated by a resistor carrying ac current I_{rms} is the same as the power dissipated by the same resistor carrying current I.

If, instead, you knew the voltage across the resistor as a function of time, never fear. If the amplitude of the voltage is V, the same reasoning we used above leads to

$$P = \frac{V_{\mathrm{rms}}^2}{R}$$

if the root-mean-square voltage is

$$V_{\mathrm{rms}} = \frac{V}{\sqrt{2}}. \tag{3.4}$$

In summary, we can do power calculations in ac circuits if we replace the amplitudes I and V with their rms values, found by dividing the amplitudes by $\sqrt{2}$. Such conversions, among peak, peak-to-peak, and rms values, are common.

Example 3.4

Convert a peak current of 3 A to rms.

Solution:

$$I_{\mathrm{rms}} = \frac{I}{\sqrt{2}} = \frac{3\text{ A}}{\sqrt{2}} = 2.12\text{ A}$$

Example 3.5

What is the amplitude of a voltage signal with an rms voltage of 6.3 V?

Solution:

$$V = \sqrt{2}V_{\mathrm{rms}} = \sqrt{2}(6.3\text{ V}) = 8.91\text{ V}$$

Power calculations are so common that, when dealing with ac, rms is the default—all quantities are rms unless otherwise specified. If a specification on some device or piece of equipment lists a voltage as “117 VAC,” this means 117 V, alternating current, rms.

Logical Check The rms value is always less than the peak value, since rms is a kind of average, and an average will *always* be less than (or equal to) the maximum. Similarly, the peak to peak value is greater than the peak value.

3.2 Reactance

3.2.1 Capacitive Reactance

Early researchers in electricity may have been surprised to find that ac would flow through a capacitor. After all, a capacitor is an open circuit, isn't it? Experiments on capacitors show that as more ac voltage is applied more current will flow. If the frequency is increased without changing the voltage, the current will increase. It can also be shown experimentally that the current through a capacitor leads the voltage by 90°, or 1/4 of a cycle. The way in which a capacitor reacts to an ac input is called the "capacitive reactance."

Let us apply a voltage to a capacitor and see what happens. Suppose the voltage is

$$v(t) = V \sin \omega t. \tag{3.5}$$

Recall Eq. (1.17), restated here:

$$i = C\frac{dv}{dt}. \tag{3.6}$$

If we insert Eq. (3.5) into Eq. (3.6), we have

$$i = CV\frac{d}{dt}\sin \omega t = CV\omega \cos \omega t = I \cos \omega t,$$

where we have defined the amplitude of the current $I = CV\omega$.

We call the ratio of these two amplitudes, V/I, the **capacitive reactance** X_C, which is a quantification of how a capacitor "reacts" to an ac input voltage:

$$X_C \equiv \frac{1}{\omega C} = \frac{V}{I}. \tag{3.7}$$

Note that this has units of resistance. We now have a new quantity which is like a resistance in some ways, but not in others.

We applied a sine wave voltage and a cosine wave current resulted. The cosine function leads sine by 90° (we say cosine leads because if you want the value of sine at some angle, you have to look 90° backward, or "earlier," on the cosine curve), so we say that in a capacitor, the *current leads the voltage* by 90°.

Example 3.6

At what frequency will a 0.0039 μF capacitor have a reactance of 33 kΩ?

Solution: Solving Eq. (3.7) for ω gives $\omega = 1/(CX_C)$. From Eq. (3.1), $\omega = 2\pi f$. Putting together the pieces, we have

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{1}{CX_C} = \frac{1}{2\pi(33\ \text{k}\Omega)(0.0039\ \mu\text{F})},$$

so $f = 1.24 \times 10^3$ Hz.

Example 3.7

If a 0.1 μF capacitor is connected across the 120 VAC, 60 Hz power line, how much current will flow?

Solution: The reactance of a 0.1 μF capacitor at a frequency of 60 Hz is

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi(60\text{ Hz})(1\times10^{-7}\text{ F})} = 2.653\times10^{4}\ \Omega.$$

Rearrange Eq. (3.7) to solve for the current:

$$I = \frac{V}{X_C} = \frac{120\text{ V}}{2.653\times10^{4}\ \Omega} = 4.52\times10^{-3}\text{ A},$$

or 4.52 mA. Remember that both the voltage in this problem and the current we found are rms values, not peak values, even though Eq. (3.7) is defined in terms of peak values. In taking the ratio, we have canceled the $\sqrt{2}$ factors.

Power When an ac voltage is applied to a resistor, the current is in phase with the voltage. The instantaneous power is

$$p = iv = (I\sin\omega t)(V\sin\omega t) = IV\sin^2\omega t.$$

Referring again to Eq. (A.10), we can write this as

$$p = \frac{IV}{2}(1-\cos 2\omega t)$$

This function is positive for all t. When an ac voltage is applied to a resistor, the power is "real." All energy flows from the generator into the load.

On the other hand, When an ac voltage is applied to a capacitor, the current leads the voltage by 90 degrees. The instantaneous power is

$$p = iv = (I\cos\omega t)(V\sin\omega t).$$

Referring again to the appendix, we can use Eq. (A.9) to write

$$p = IV\left\{\frac{1}{2}\left[\sin(\omega t-\omega t)+\sin(\omega t+\omega t)\right]\right\} = \frac{IV}{2}\sin 2\omega t.$$

We're left with a single power of sine, which oscillates with double the frequency of either the input voltage or the resulting current. Note that it has equal parts above and below the horizontal axis, so power is taken from the circuit for half of every cycle, and given back to the circuit during the other half. The net power averaged over time is zero.

Here we have an incongruous situation. There is a voltage applied to a circuit element (capacitor) and there is a current flowing through it, but the power is zero! This can be verified experimentally. It must have been fascinating for the first person to observe it.

3.2.2 Inductive Reactance

It was potentially surprising for early researchers that a a coil of wire presented more "resistance" to the flow of alternating current than did the identical length of wire which was not wound into a coil. Experiments on inductors show that as more ac voltage is applied, more current will flow. If the frequency is increased without changing the voltage, the current will decrease. It can also be shown experimentally that the current through an inductor lags the voltage across it by 90°, or 1/4 of a cycle. The way in which an inductor reacts to an ac signal is called the "inductive reactance."

Let's apply a current to an inductor and see what happens. Assume a current of the form

$$i(t) = I \sin \omega t. \tag{3.8}$$

Apply the $i - v$ relationship for the inductor [Eq. (1.20)]:

$$v = L\frac{di}{dt}. \tag{3.9}$$

Substitute Eq. (3.8) into Eq. (3.9), and take the derivative:

$$v = IL\frac{d}{dt} \sin \omega t = IL\omega \cos \omega t = V \cos \omega t,$$

where we have defined the voltage amplitude to be $V \equiv IL\omega$.

As before, we define the ratio of the two amplitudes V/I, a measure of the way an inductor "reacts" to an ac input, as the **inductive reactance**:

$$X_\text{L} \equiv \omega L = \frac{V}{I}. \tag{3.10}$$

Like capacitive reactance, inductive reactance has units of resistance and behaves in some, though not all, ways like a resistance

In this case, we applied a sine wave current and got a cosine wave voltage. The cosine function leads sine by 90°, so the voltage across an inductor leads the current through it by 90°. Alternatively, the *current lags the voltage* by 90°.

Example 3.8

What is the value of an inductor which will present a reactance of 47 kΩ at a frequency of 100 kHz?

Solution: Solving Eq. (3.10) for L and substituting Eq. (3.1), we have

$$L = \frac{X_\text{L}}{\omega} = \frac{X_\text{L}}{2\pi f} = \frac{47 \times 10^3\ \Omega}{2\pi(100 \times 10^3\ \text{Hz})} = 7.48 \times 10^{-2}\ \text{H},$$

or 74.8 mH.

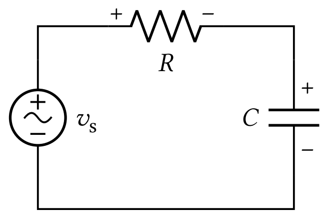


Figure 3.2: The series *RC* circuit with an ac source.

Example 3.9

If a 500 μH inductor is used in a power line (mains electricity) filter, how much voltage drop will be introduced at a current of 25 A?

Solution: The power line frequency in the US is 60 Hz. Then

$$X_L = \omega L = 2\pi f L = 2\pi(60\text{ Hz})(500\ \mu\text{H}) = 0.1885\ \Omega$$

The voltage drop is $V = IX_L = (25\text{ A})(0.1885\ \Omega) = 4.71$ V.

The power consumed by an ideal inductor is zero by the same argument as for capacitors. It makes no difference that the current is lagging instead of leading. The phase difference is still 90°.

3.3 The Series *RC* Circuit

We saw the series *RC* circuit before, back in Section 2.1.3. We examine here a slightly modified version, shown in Fig. 3.2, with an ac voltage source.

3.3.1 The Real Solution

We will first solve this in the normal way. Applying KVL gives

$$v_s = v_R + v_c = iR + v_c.$$

Since this is a series circuit, we know that the current i is the same everywhere, so the current through the resistor can be related to the voltage across the capacitor by Eq. (1.17),

$$i = C\frac{dv_c}{dt},$$

to get

$$\frac{dv_c}{dt} + \frac{1}{RC}v_c = \frac{1}{RC}v_s, \tag{3.11}$$

where we have also rearranged and divided through by RC. Now we need to decide what kind of ac voltage we're applying. Sinusoidal functions are most common, so we'll take

$$v_s(t) = V_s\cos\omega t, \tag{3.12}$$

so Eq. (3.11) becomes

$$\frac{dv_c}{dt} + \frac{1}{RC}v_c = \frac{V_s}{RC}\cos\omega t. \tag{3.13}$$

As usual, our first step is to find the particular solution. A constant does not work now, since the right-hand side depends on time. Alternatively, you might guess something of the form $v_p = A\cos\omega t$. Plugging it in gives

$$-A\omega\sin\omega t + \frac{A}{RC}\cos\omega t = \frac{V_s}{RC}\cos\omega t.$$

This is no good: since sine and cosine are linearly independent, there is no constant value of A which can satisfy this. On the other hand, sine and cosine *can* be turned into each other through phase shifts. So let's guess a solution of the form $v_p = A\cos(\omega t + \phi)$. Substitution yields

$$-A\omega\sin(\omega t + \phi) + \frac{A}{RC}\cos(\omega t + \phi) = \frac{V_s}{RC}\cos\omega t.$$

Refer to Eqs. (A.1) and (A.2) to write

$$-A\omega\,(\sin\omega t\cos\phi + \cos\omega t\sin\phi) + \frac{A}{RC}(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = \frac{V_s}{RC}\cos\omega t$$

$$\left(-A\omega\cos\phi - \frac{A}{RC}\sin\phi\right)\sin\omega t + \left(-A\omega\sin\phi + \frac{A}{RC}\cos\phi\right)\cos\omega t = \frac{V_s}{RC}\cos\omega t.$$

By comparing coefficients of sine and cosine on the two sides, we can conclude that

$$A\omega\cos\phi + \frac{A}{RC}\sin\phi = 0 \tag{3.14a}$$

$$-A\omega\sin\phi + \frac{A}{RC}\cos\phi = \frac{V_s}{RC} \tag{3.14b}$$

From Eq. (3.14a), we have

$$\sin\phi = -\omega RC\cos\phi \qquad \Longrightarrow \qquad \phi = \arctan(-\omega RC) = -\arctan(\omega RC). \tag{3.15}$$

Plugging the first form into Eq. (3.14b) gives

$$A\left[-\omega\,(-\omega RC) + \frac{1}{RC}\right]\cos\phi = \frac{V_s}{RC} = A\left[\frac{1 + (\omega RC)^2}{RC}\right]\cos\phi.$$

From trigonometry, $\cos(\arctan x) = 1/\sqrt{1 + x^2}$, so, using the last expression in Eq. (3.15), we have

$$\cos[-\arctan(\omega RC)] = \frac{1}{\sqrt{1 + (\omega RC)^2}}, \tag{3.16}$$

where we have used the fact that cosine is an even function. Then

$$A = \frac{V_s}{\sqrt{1 + (\omega RC)^2}}$$

In conclusion, the particular solution for the capacitor voltage in the series *RC* circuit is

$$v_p(t) = \frac{V_s}{\sqrt{1 + (\omega RC)^2}}\cos[\omega t - \arctan(\omega RC)]. \tag{3.17}$$

The homogeneous equation solution is exactly the same as it was in Section 2.1.3 [cf. Eq. (2.8)]. If the capacitor starts out with no charge and no input at time $t = 0$, then we have

$$v_\mathrm{h}(t) = Be^{-t/\tau},$$

where B is a constant determined by initial conditions.

To get the general solution, we add together the particular and homogeneous solutions to get

$$v_\mathrm{c}(t) = Be^{-t/\tau} + \frac{V_\mathrm{s}}{\sqrt{1+(\omega RC)^2}}\cos[\omega t - \arctan(\omega RC)]. \tag{3.18}$$

For example, suppose that the voltage source is initially off (and the capacitor is uncharged) and then it is switched on at time $t = 0$. Initially, the capacitor has no charge, so the voltage across it is zero:

$$v_\mathrm{c}(0) = 0 = Be^0 + \frac{V_\mathrm{s}}{\sqrt{1+(\omega RC)^2}}\cos\left[0 - \arctan(\omega RC)\right] = B + \frac{V_\mathrm{s}}{1+(\omega RC)^2},$$

where we have used Eq. (3.16). Solve this for B and substitute it into Eq. (3.18):

$$v_\mathrm{c}(t) = -\frac{V_\mathrm{s}}{1+(\omega RC)^2}e^{-t/\tau} + \frac{V_\mathrm{s}}{\sqrt{1+(\omega RC)^2}}\cos\left[\omega t - \arctan(\omega RC)\right].$$

The first term is a decaying exponential—and it decays quite rapidly in most cases, since usually $\tau = RC$ is a small number. The other term continues to oscillate forever (or until we disconnect the voltage source), so that's really the more interesting term.

One lesson to draw from this is that, with most sinusoidal ac circuits, the particular solution contains the important information, while the homogeneous solution tells you about the **transient behavior**, or what happens when the circuit is first connected, before it settles into steady-state.

3.3.2 Phasors; the Complex (but Simpler) Solution

In this section (and in many to come) we will make extensive use of complex numbers. If you are not familiar with them, read Appendix A.3 before continuing.

Now consider the same circuit, with the same driving voltage $v_\mathrm{s}(t) = V_\mathrm{s}\cos\omega t$. But let's look at this differently. By Euler's formula, $e^{j\theta} = \cos\theta + j\sin\theta$, where $j = \sqrt{-1}$ is the imaginary unit (we don't use i in electronics or electrical engineering to avoid confusion with current). The cosine part is the real part of $e^{j\theta}$, so we can write the source voltage as

$$v_\mathrm{s}(t) = \Re\left[V_\mathrm{s}e^{j\omega t}\right].$$

We will assume that the solution for the voltage across the resistor has the form

$$v_\mathrm{c}(t) = \Re\left[\hat{V}_\mathrm{c}e^{j\omega t}\right],$$

where $\hat{V}_\mathrm{c}$ is a complex number called the *complex amplitude* or the **phasor** representing the voltage across the capacitor. It encapsulates the time-invariant information about that voltage, including the amplitude and the phase. In polar coordinates, we would write a phasor as (in this case) $\hat{V}_\mathrm{c} = V_\mathrm{c}e^{j\phi}$, where $V_\mathrm{c} = |\hat{V}_\mathrm{c}|$ is the *magnitude* of the phasor, and ϕ is the *phase* of the phasor. The time-dependence is left outside of the phasor, in the factor of $e^{j\omega t}$.

Return now to the differential equation that resulted from applying KVL around the *RC* circuit, and plug in the complex forms (both the phasor and the time factor) for v_c and v_s. We'll take the real part at the end.

$$\frac{d}{dt}(\hat{V}_c e^{j\omega t}) + \frac{1}{RC}\hat{V}_c e^{j\omega t} = \frac{1}{RC}V_s e^{j\omega t} = j\omega \hat{V}_c e^{j\omega t} + \frac{1}{RC}\hat{V}_c e^{j\omega t}.$$

Canceling the exponential and factoring leaves

$$\hat{V}_c\left(j\omega + \frac{1}{RC}\right) = \frac{1}{RC}V_s.$$

Isolate $\hat{V}_c$, and multiply top and bottom by RC to get

$$\hat{V}_c = \frac{V_s}{1 + j\omega RC}.$$

Now we have a complex number in the denominator, which is bad. To get it into the numerator, multiply top and bottom by the complex conjugate, $1 - j\omega RC$, to get

$$\hat{V}_c = \frac{V_s - j\omega RC V_s}{1 + (\omega RC)^2} = \frac{V_s}{1 + (\omega RC)^2} + j\frac{-\omega RC V_s}{1 + (\omega RC)^2}. \tag{3.19}$$

The magnitude of the phasor is given by the Pythagorean theorem (more on this later):

$$V_c = \sqrt{\left(\frac{V_s}{1 + (\omega RC)^2}\right)^2 + \left(\frac{\omega RC V_s}{1 + (\omega RC)^2}\right)^2} = \frac{\sqrt{V_s^2 + (\omega RC V_s)^2}}{1 + (\omega RC)^2} = V_s\frac{\sqrt{1 + (\omega RC)^2}}{1 + (\omega RC)^2}$$

$$V_c = \frac{V_s}{\sqrt{1 + (\omega RC)^2}}$$

Look at that! Compare that to the coefficient of the cosine term (remember: that's the one we care about) in Eq. (3.18). They're the same, and this was much simpler.

Now onto the phase. The phase of our current complex number is given by

$$\phi = \arctan\left(\frac{\Im(\hat{V}_c)}{\Re(\hat{V}_c)}\right).$$

Considering the final equality in Eq. (3.19), we can see that both the real part (the first term) and the imaginary part (everything but the j in the second term) have the same denominator, so they will cancel upon division. Likewise, the V_s in the numerator of each will cancel, leaving us with

$$\phi = \arctan\left(\frac{-\omega RC}{1}\right) = \arctan(-\omega RC) = -\arctan(\omega RC),$$

exactly as we found before!

Putting together the magnitude and the phase, we find

$$\hat{V}_c = V_c e^{j\phi} = \frac{V_s}{\sqrt{1 + (\omega RC)^2}} e^{-j\arctan(\omega RC)}.$$

The full, complex representation of our result is

$$\hat{V}_c e^{j\omega t} = \frac{V_s}{\sqrt{1 + (\omega RC)^2}} e^{j[\omega t - \arctan(\omega RC)]}.$$

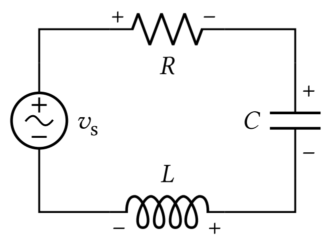


Figure 3.3: The series *RLC* circuit, with polarities for the elements marked.

Finally, we take the real part to get

$$v_c(t) = \frac{V_s}{\sqrt{1 + (\omega RC)^2}} \cos[\omega t - \arctan(\omega RC)]. \tag{3.20}$$

Thus, using complex exponentials, we can capture the part of the solution that is most important—the part that does not decay quickly away—with much less effort than if we were using trigonometric functions.

3.4 Phasors and the Impedance Model

3.5 The Series *RLC* Circuit

We turn now to the series *RLC* circuit. Fair warning, we are going to give the trig-identity appendix sections a good workout here. But there's light at the end of the tunnel: we're going to find that, in most cases, we won't have to deal with so much algebraic and trigonometric tediousness.

Consider the diagram shown in Fig. 3.3. By KVL,

$$v_s = v_R + v_c + v_L. \tag{3.21}$$

The next step, as usual, is to plug in the element relationships. There are several ways to do this. We will choose the current i in the circuit as the variable for which we want to solve. We can use Ohm's law [Eq. (1.10)] and the element relationships for capacitors [Eq. (1.17)] and inductors [Eq. (1.20)], along with a little basic calculus, to write

$$v_s = Ri + \int \frac{i}{C}\,dt + L\frac{di}{dt}.$$

Take the time-derivative of the equation and use the fundamental theorem of calculus to find

$$\frac{dv_s}{dt} = R\frac{di}{dt} + \frac{1}{C}i + L\frac{d^2 i}{dt^2}.$$

Rearrange this to get it into a standard form:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}\frac{dv_s}{dt}. \tag{3.22}$$

Define a couple of convenience variables whose convenience will become more apparent as we move forward:

$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad \text{and} \qquad \gamma \equiv \frac{R}{2L}, \tag{3.23}$$

so that Eq. (3.22) becomes

$$\frac{d^2 i}{dt^2} + 2\gamma \frac{di}{dt} + \omega_0^2 i = \frac{1}{L}\frac{dv_s}{dt}. \tag{3.24}$$

3.5.1 The dc Switching Circuit

Let's first take our source to be a dc voltage source, so that

$$v(s) = \begin{cases} 0 & t < 0 \\ V_s & t \geq 0 \end{cases} \tag{3.25}$$

During both time intervals, we have a constant value for v_s, so Eq. (3.24) becomes

$$\frac{d^2 i}{dt^2} + 2\gamma \frac{di}{dt} + \omega_0^2 i = 0. \tag{3.26}$$

This is already a homogeneous solution, so the particular solution is automatically $i_p = 0$. We're left with only the homogeneous solution, so we can drop the subscript h. Since this is a second-order differential equation, the general solution must be a linear combination of two linearly-independent functions. Guess initially that $i = Ae^{\xi t}$, so that

$$A\xi^2 e^{\xi t} + 2A\xi\gamma e^{\xi t} + \omega_0^2 A e^{\xi t} = 0.$$

Canceling yields

$$\xi^2 + 2\gamma\xi + \omega_0^2 = 0.$$

By the quadratic formula,

$$\xi = \frac{(-2\gamma) \pm \sqrt{(2\gamma)^2 - 4\omega_0^2}}{2} = -\gamma \pm \frac{2\sqrt{\gamma^2 - \omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

Appendix A

Mathematical References

A.1 Trigonometric Identities

A.1.1 Shifts

Function	Period	Quarter-Period Shift	Half-Period Shift
sin	2π	$\sin\left(\phi \pm \frac{\pi}{2}\right) = \pm\cos\phi$	$\sin(\phi + \pi) = -\sin\phi$
cos	2π	$\cos\left(\phi \pm \frac{\pi}{2}\right) = \mp\sin\phi$	$\cos(\phi + \pi) = -\cos\phi$
tan	π	$\tan\left(\phi \pm \frac{\pi}{4}\right) = \frac{\tan\phi \pm 1}{1 \mp \tan\phi}$	$\tan\left(\phi + \frac{\pi}{2}\right) = -\cot\phi$
csc	2π	$\csc\left(\phi \pm \frac{\pi}{2}\right) = \pm\sec\phi$	$\csc(\phi + \pi) = -\csc\phi$
sec	2π	$\sec\left(\phi \pm \frac{\pi}{2}\right) = \mp\csc\phi$	$\sec(\phi + \pi) = -\sec\phi$
cot	π	$\cot\left(\phi \pm \frac{\pi}{4}\right) = \frac{\cot\phi \pm 1}{1 \mp \cot\phi}$	$\cot\left(\phi + \frac{\pi}{2}\right) = -\tan\phi$

A.1.2 Angle Sum and Difference Formulae

$$\sin(\phi \pm \theta) = \sin\phi\cos\theta \pm \cos\phi\sin\theta \tag{A.1}$$

$$\cos(\phi \pm \theta) = \cos\phi\cos\theta \mp \sin\phi\sin\theta \tag{A.2}$$

$$\tan(\phi \pm \theta) = \frac{\tan\phi \pm \tan\theta}{1 \mp \tan\phi\tan\theta} \tag{A.3}$$

A.1.3 Sum, Difference, and Product Formulae

$$\sin\theta \pm \sin\phi = 2\sin\left(\frac{\theta \pm \phi}{2}\right)\cos\left(\frac{\theta \mp \phi}{2}\right) \tag{A.4}$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right) \tag{A.5}$$

$$\cos\theta - \cos\phi = 2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\phi - \theta}{2}\right) \tag{A.6}$$

$$\sin\theta\sin\phi = \frac{1}{2}\left[\cos(\theta - \phi) - \cos(\theta + \phi)\right] \tag{A.7}$$

$$\cos\theta\cos\phi = \frac{1}{2}\left[\cos(\theta - \phi) + \cos(\theta + \phi)\right] \tag{A.8}$$

$$\sin\theta\cos\phi = \frac{1}{2}\left[\sin(\theta - \phi) + \sin(\theta + \phi)\right] \tag{A.9}$$

A.1.4 Power Reduction Formulae

$$\sin^2 \phi = \frac{1}{2}(1 - \cos 2\phi) \tag{A.10}$$

$$\cos^2 \phi = \frac{1}{2}(1 + \cos 2\phi) \tag{A.11}$$

$$\sin^3 \phi = \frac{1}{4}(3 \sin \phi - \sin 3\phi) \tag{A.12}$$

$$\cos^3 \phi = \frac{1}{4}(3 \cos \phi + \cos 3\phi) \tag{A.13}$$

$$\sin^4 \phi = \frac{1}{8}(3 - 4\cos 2\phi + \cos 4\phi) \tag{A.14}$$

$$\cos^4 \phi = \frac{1}{8}(3 + 4\cos 2\phi + \cos 4\phi) \tag{A.15}$$

A.1.5 Multiple-Angle Formulae

Double-Angle

$$\sin 2\phi = 2 \sin \phi \cos \phi = \frac{2 \tan \phi}{1 + \tan^2 \phi} \tag{A.16}$$

$$\cos 2\phi = 2\cos^2 \phi - 1 = 1 = 2\sin^2 \phi = \cos^2 \phi - \sin^2 \phi = \frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} \tag{A.17}$$

$$\tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi} \tag{A.18}$$

Half-Angle

$$\sin \frac{\phi}{2} = a\sqrt{\frac{1 - \cos \phi}{2}} \tag{A.19}$$

$$\cos \frac{\phi}{2} = b\sqrt{\frac{1 + \cos \phi}{2}} \tag{A.20}$$

$$\tan \frac{\phi}{2} = c\sqrt{\frac{1 - \cos \phi}{1 + \cos \phi}} \tag{A.21}$$

where

$$a = \begin{cases} +1 & \text{if } \phi \text{ is in quadrant I or II} \\ -1 & \text{if } \phi \text{ is in quadrant II or IV} \end{cases} \tag{A.22}$$

$$b = \begin{cases} +1 & \text{if } \phi \text{ is in quadrant I or IV} \\ -1 & \text{if } \phi \text{ is in quadrant II or III} \end{cases} \tag{A.23}$$

$$c = \begin{cases} +1 & \text{if } \phi \text{ is in quadrant I or III} \\ -1 & \text{if } \phi \text{ is in quadrant II or IV} \end{cases} \tag{A.24}$$

Triple-Angle and Quadruple-Angle

$$\sin 3\phi = 3\sin\phi - 4\sin^3\phi \tag{A.25}$$

$$\cos 3\phi = 4\cos^3\phi - 3\cos\phi \tag{A.26}$$

$$\tan 3\phi = \frac{3\tan\phi - \tan^3\phi}{1 - 3\tan^2\phi} \tag{A.27}$$

$$\sin 4\phi = 4\sin\phi\cos\phi - 8\sin^3\phi\cos\phi \tag{A.28}$$

$$\cos 4\phi = 8\cos^4\phi - 8\cos^2\phi + 1 \tag{A.29}$$

$$\tan 4\phi = \frac{4\tan\phi - 4\tan^3\phi}{1 - 6\tan^2\phi + \tan^4\phi} \tag{A.30}$$

A.1.6 Relations Between Inverse Trigonometric Functions

$$\arcsin(-\phi) = -\arcsin\phi \tag{A.31}$$

$$\arccos(-\phi) = \pi - \arccos\phi \tag{A.32}$$

$$\arctan(-\phi) = -\arctan\phi \tag{A.33}$$

$$\arcsin\phi + \arccos\phi = \frac{\pi}{2} \tag{A.34}$$

A.2 Properties of Exponentials

$$a^x a^y = a^{x+y} \tag{A.35}$$

$$\frac{a^x}{a^y} = a^{x-y} \tag{A.36}$$

$$\left(a^x\right)^y = a^{xy} \tag{A.37}$$

$$(ab)^x = a^x b^x \tag{A.38}$$

$$a^{-x} = \frac{1}{a^x} \tag{A.39}$$

$$e^{j\theta} = \sin\theta + j\sin\theta \tag{A.40}$$

where $a, b \in \mathbb{R} > 0$; $x, y \in \mathbb{R}$; e is the base of the natural logarithm; and $j = \sqrt{-1}$ is the imaginary unit.

A.3 Complex Numbers

If $a, b \in \mathbb{R}$, we can write a **complex number** as

$$\hat{z} = a + jb, \tag{A.41}$$

where $j \equiv \sqrt{-1}$ is the imaginary unit[1]. We say that a is the real part of $\hat{z}$, and b is the imaginary part:

$$a = \Re(\hat{z}) \qquad \text{and} \qquad b = \Im(\hat{z}). \tag{A.42}$$

The **complex conjugate** of a complex number is the same complex number but with the imaginary part negated:

$$\hat{z}^* = a - jb. \tag{A.43}$$

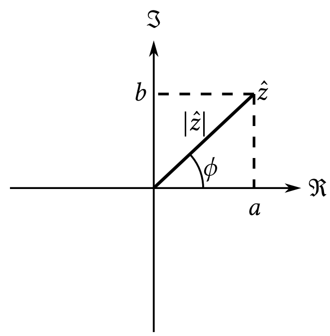


Figure A.1: A complex number drawn in the complex plane. The real part is plotted on the horizontal axis, and the imaginary part is shown on the vertical axis.

A common way of visualizing complex numbers is by drawing them as vectors in the complex plane, shown in Fig. A.1. The horizontal axis shows the real part and the vertical axis gives the imaginary part.

This can be written in polar form as

$$\hat{z} = ze^{j\phi} = z\left(\cos\phi + j\sin\phi\right), \tag{A.44}$$

where the second equality follows from Euler's famous formula and

$$z = |\hat{z}| = \sqrt{\hat{z}\hat{z}^*} = \sqrt{a^2 + b^2} \tag{A.45}$$

is the **modulus** or magnitude of the complex number and

$$\phi = \text{arctan2}(b, a) \tag{A.46}$$

is the **phase angle** or the **argument**. The meaning of the argument is clear from the complex-plane depiction (Fig. A.1). In case you are not familiar with arctan2, it is similar to arctan, but it gives you an angle in the correct quadrant. If ϕ is in the first quadrant,

$$\text{arctan2}(b, a) = \arctan\left(\frac{b}{a}\right). \tag{A.47}$$

However, if ϕ is in the third quadrant, regular arctan yields an angle in the first quadrant anyway. You can use regular arctan, but you will have to manually adjust the angle with some πs.

Some Complex-Number Identities

$$\hat{z}_1\hat{z}_2 = \left(z_1 e^{j\phi_1}\right)\left(z_2 e^{j\phi_2}\right) = z_1 z_2 e^{j(\phi_1+\phi_2)} \tag{A.48}$$

$$\frac{\hat{z}_1}{\hat{z}_2} = \frac{z_1}{z_2} e^{j(\phi_1-\phi_2)} \tag{A.49}$$

$$\hat{z}^x = \left(ze^{j\phi}\right)^x = z^x e^{jx\phi} \tag{A.50}$$

[1] We use j instead of i in electronics to avoid confusion with current

A.4 Taylor Series

The Taylor series for a function $f(x)$ of one variable x about some point $x = a$ is

$$f(x) = f(a) + (x-a)\left.\frac{df}{dx}\right|_{x=a} + \frac{1}{2!}(x-a)^2\left.\frac{d^2f}{dx^2}\right|_{x=a} + \frac{1}{3!}(x-a)^3\left.\frac{d^3f}{dx^3}\right|_{x=a} + \cdots \tag{A.51}$$

For a function $f(x, y)$ of two variables x and y about points $x = a$ and $y = b$, we have

$$\begin{aligned} f(x,y) =& f(a,b) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a,y=b} + (y-b)\left.\frac{\partial f}{\partial y}\right|_{x=a,y=b} \\ &+ \frac{1}{2!}\left[(x-a)^2\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a,y=b} + 2(x-a)(y-b)\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x=a,y=b} + (y-b)^2\left.\frac{\partial^2 f}{\partial y^2}\right|_{x=a,y=b}\right] \\ &+ \cdots \end{aligned} \tag{A.52}$$

Some other commonly used series expansions that derive from Taylor's result are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{A.53}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \tag{A.54}$$

$$\ln(x) = \left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x-1}{x}\right)^n \tag{A.55}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} \tag{A.56}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^{n}\frac{x^{2n}}{(2n)!}. \tag{A.57}$$

Conclude with the binomial expansion

$$(a+x)^n = a^n + na^{n-1}x + \frac{1}{2!}n(n-1)a^{n-2}x^2 + \frac{1}{3!}n(n-1)(n-2)a^{n-3}x^3 + \cdots. \tag{A.58}$$